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Fixed Point Theorems for ∈ − Chainable Fuzzy Metric Space

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Abstract

In this paper, we present fixed point theorems for six weakly compatible mappings in a complete ∈ −chainable fuzzy metric space, without requiring any of the mappings to be continuous. Our findings broaden and enhance several established results in fixed-point theory across various spaces.

Keywords

Fuzzy Metric Space, $\varepsilon -$ Chainable Fuzzy Metric Space, Weakly Compatible Mappings, Common Fixed Point.

1 Introduction

The origin of fuzzy set theory and fuzzy mathematics traces back to Zadeh [15], who introduced the concept of fuzzy sets in 1965 as a means to represent imprecision and uncertainty in real-world phenomena. Since its inception, the theory has found widespread applications across numerous scientific and engineering disciplines, including neural networks, stability analysis, mathematical programming, genetics, neuroscience, image processing, and control systems.One of the foundational tools in dealing with such applications is fixed point theory, which has significantly contributed to the evolution and extension of various analytical and topological concepts within the fuzzy framework. In 1975, Kramosil and Michalek [8] developed the idea of a fuzzy metric space, extending the probabilistic metric space concept into a fuzzy context. This notion was further refined in 1994 by George and Veeramani [3], who provided a more rigorous formulation of fuzzy metric spaces.Building on this foundation, Grabiec [4] presented a fuzzy version of Banach's fixed point theorem in 1988. Around the same time, Sessa [9] introduced the concept of weakly commuting mappings to enhance the classical commutativity conditions in fixed point theorems. Jungck [5,7] contributed significantly by defining the concept of compatibility and proving common fixed point theorems for set-valued functions, even in the absence of continuity. Later, in 2006, Jungck and Rhoades [6] extended this idea further with the notion of weakly compatible mappings, generalizing the previous compatibility definitions. In the context of fuzzy metric spaces, Cho [2] introduced the notion of compatible mappings in 1997. Subsequently, Vasuki [14] proposed the idea of R–weakly commuting mappings and demonstrated a fixed point theorem based on this concept. Singh and Chauhan [12] expanded the theory by incorporating the concept of compatibility within fuzzy metric spaces, while Singh and Jain [13] explored semi-compatibility and weak compatibility of mappings in such spaces. Further developments were made by Sharma and Deshpande [10], who proved fixed point theorems for multiple discontinuous and non-compatible mappings in non-complete fuzzy metric spaces. They later extended this work to intuitionistic fuzzy metric spaces continuing the exploration of fixed point results under broader conditions. In this paper, we establish fixed point theorems for six weakly compatible self-mappings defined on a complete, chainable fuzzy metric space, without assuming continuity. Our results serve to generalize and extend many existing theorems in fixed point literature across a variety of metric settings.

2 Preliminaries

Definition 2.1: A 3 – tuple $(X, \mathcal{M}, *)$ is called a $\mathcal{M}-$ fuzzy metric space if X is an arbitrary (non - empty) set, $*$ is a continuous $t-norm$, and $\mathcal{M}$ is a fuzzy set on $X^2 \times (0,\infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,

(i) $\mathcal{M}(x,y,t) = 0$,
(ii) $\mathcal{M}(x,y,t) = 1$ for all $t > 0$ if and only if $x = y$,
(iii) $\mathcal{M}(x,y,t) = \mathcal{M}(y,x,t)$,
(iv) $\mathcal{M}(x,y,t) * \mathcal{M}(y,z,s) \leq \mathcal{M}(x,z,t+s)$,
(v) $\mathcal{M}(x,y,\ .) : [0,1] \to [0,1]$ is left continuous.

Example 2.1: Let X be the subset of $\mathbb{R}^2$ defined by $X = \{A, B, C, D, E\}$, where $A = (0,0)$, $B = (1,0), C = (1,2), D = (0,1), E = (1,3)$, $\varphi(\tau) = 1 - \sqrt{\tau}$ for all $\tau \in [0,1]$ and $\mathcal{M}(x,y,t) = e^{-\frac{2d(x,y)}{t}}$ for all $t > 0$, where $d(x,y)$ denotes the Euclidean distance of $\mathbb{R}^2$. Clearly, $(X, \mathcal{M}, *)$ is an $\mathcal{M}-$ complete fuzzy metric space with respect to the $t-$ norm:

$$a * b = ab.$$

Example 2.2: Let (X, d) be a metric space. Define $a * b = ab$, or $a * b = \min(a, b)$, and for all x, y and $t > 0$,

$$\mathcal{M}(x,y,t) = \frac{t}{t + d(x,y)}$$

then $(X, \mathcal{M}, *)$ is a fuzzy metric space. We call this fuzzy metric $\mathcal{M}$ induced by the metric d, the standard fuzzy metric.

Lemma 2.1: $\mathcal{M}(x,y,.)$ is non-decreasing for all $x, y \in X$.

Proof: Suppose $\mathcal{M}(x,y,t) > M(x,y,s)$ for some $0 < t < s$. then $\mathcal{M}(x,y,t) * \mathcal{M}(y,y,s-t) \leq \mathcal{M}(x,y,s) < M(x,y,t)$.
Since $\mathcal{M}(y,y,s-t) = 1$,

therefore, $\mathcal{M}(x,y,t) \leq \mathcal{M}(x,y,s) < M(x,y,t)$, which is a contradiction. Thus, $\mathcal{M}(x,y,.)$ is non-decreasing for all $x, y \in X$.

Definition 2.2: Let $(X, \mathcal{M}, *)$ be a fuzzy metric space:

(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$, if $\lim_{n\to\infty} \mathcal{M}(x_n, x, t) = 1$, for all $t > 0$.

(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if

$$\lim_{n \to \infty} \mathcal{M}\left(x_{n+p}, x_n, t\right) = 1, \text{ for all } t > 0 \text{ and } p > 0.$$

(iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Remark 2.1.: Since $*$ is continuous, if follows from the condition (iv) of **Definition 2.1** that the limit of the sequence in fuzzy metric space is uniquely determined.

Let $(X, \mathcal{M}, *)$ be a fuzzy metric space with the following condition :

$\lim_{t \to \infty} \mathcal{M}(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$

Lemma 2.2: If for all $x, y \in X$, $t > 0$ and $0 < k < 1$, $\mathcal{M}(x, y, kt) \geq \mathcal{M}(x, y, t)$, then $x = y$.

Proof: Suppose that there exists $0 < k < 1$ such that

$\mathcal{M}(x, y, kt) \geq \mathcal{M}(x, y, t)$ for all $x, y \in X$ and $t > 0$.

Then, $\mathcal{M}(x, y, t) \geq \mathcal{M}\left(x, y, \frac{t}{k}\right)$,

and so $\mathcal{M}(x, y, t) \geq \mathcal{M}\left(x, y, \frac{t}{k^n}\right)$ for positive integer n.

Taking limit as $n \to \infty$,

$\mathcal{M}(x, y, t) \geq 1$ and hence $x = y$.

Lemma 2.3: Let $(X, \mathcal{M}, *)$ be a fuzzy metric space and

$\{y_n\}$ be a sequence in X. If there exists a number $k \in (0, 1)$ such that

$$\mathcal{M}(y_{n+2}, y_{n+1}, kt) \geq \mathcal{M}(y_{n+1}, y_n, t),$$

for all $t > 0$ and $n = 1, 2, \ldots,$

then $\{y_n\}$ is a Cauchy sequence in X.

Definition 2.3: Let A and B be mappings from a fuzzy metric space $(X, \mathcal{M}, *)$ into itself. The mappings A and B are said to be compatible if

$$\lim_{n \to \infty} \mathcal{M}(ABx_n, BAx_n, t) = 1, \text{ for all } t > 0,$$

Whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \to \infty} A x_n = \lim_{n \to \infty} B x_n = z \quad \text{for some } z \in X.$$

Definition 2.4: Two self mappings A and B of a fuzzy metric space $(X, \mathcal{M}, *)$ are said to be weakly compatible if $ABu = BAu$ whenever $Au = Bu$ for some

$u \in X$. If the self mappings A and B of a fuzzy metric space $(X, \mathcal{M}, *)$ are compatible, then they are weakly compatible, but the converse is not necessarily true.

Example 2.3: Let $X = [0, 4]$ and $a * b = \min\{a, b\}$. Let $\mathcal{M}$ be the standard fuzzy metric induced by d, where $d\,(x, y) = |x - y|$ for $x, y \in X$. Define two self mappings A and B of the fuzzy metric space $(X, \mathcal{M}, *)$ by:

$$Ax = \begin{cases} 4 - x, & 0 \le x \le 2 \\ 4, & 2 \le x \le 4 \end{cases}$$

$$Bx = \begin{cases} x, & 0 \le x \le 2 \\ 4, & 2 \le x \le 4 \end{cases}$$

Let $\{x_n\} = \{1 - (1/n)\}$. Then it can be easily proved that the self mappings A and B are weakly compatible but they are not compatible.

Definition 2.5: A finite sequence $x = x_0, x_1, \cdots, x_n = y$ in a fuzzy metric space $(X, \mathcal{M}, *)$ is called $\varepsilon -$ chain from x to y if there exists $\varepsilon > 0$ such that $\mathcal{M}\,(x_i, x_{i-1}, t) > 1 - \varepsilon$ for all $t > 0$ and $i = 1, 2, \cdots, n$.

A fuzzy metric space $(X, \mathcal{M}, *)$ is called $\varepsilon -$ chainable if there exists an $\varepsilon -$ chain from x to y, for any $x, y \in X$.

3 The Main Results

Theorem 3.1: Let $(X, \mathcal{M}, *)$ be a complete $\varepsilon -$ chainable fuzzy metric space and let A, B, S, T, P and Q be the self mappings of X, satisfying the following conditions:

(1) $A(X) \subset ST(X)$ and $B(X) \subset PQ(X)$;
(2) The pair (A, PQ) and (B, ST) are weakly compatible ;
(3) There exists a constant $k \in (0, 1)$, such that for every $x, y \in X$ and $t > 0$,

$$\mathcal{M}\,(Ax, By, kt) \ge \{\mathcal{M}\,(PQx, STy, t) * \mathcal{M}\,(Ax, PQx, t) * \mathcal{M}\,(By, STy, t) * \mathcal{M}\,(Ax, STy, t) * \mathcal{M}\,(By, PQx, t)\}.$$

Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof: We can find a Cauchy sequence $\{y_n\}$ in X such that

$y_{2n-1} = STx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = PQx_{2n} = Bx_{2n-1}$ for $n = 1, 2, 3, \cdots$ From completeness, $y_n \to z$ for some $z \in X$, and so $\{Ax_{2n-2}\}, \{PQx_{2n}\}$, $\{Bx_{2n-1}\}$ and $\{STx_{2n-1}\}$ also converge to z. Similarly we can show that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, hence there exists $z \in X$ such that $\{x_n\}$ converge to z. Hence there exists $u, v \in X$ such that $PQu = z$ and $STv = z$ respectively. By (3), we have

$$\mathcal{M}\,(Au, y_{2n}, kt) = \mathcal{M}\,(Au, Bx_{2n-1}, kt) \ge \{\mathcal{M}\,(PQu, STx_{2n-1}, t) * \mathcal{M}\,(Au, PQu, t) * \mathcal{M}\,(Bx_{2n-1}, STx_{2n-1}, t) * \mathcal{M}\,(Au, STx_{2n-1}, t) * \mathcal{M}\,(Bx_{2n-1}, PQu, t)\}.$$

Taking the limit as $n \to \infty$,

$$\mathcal{M}\,(Au, z, kt) \ge \{\mathcal{M}\,(z, z, t) * \mathcal{M}\,(Au, z, t) * \mathcal{M}\,(z, z, t) * \mathcal{M}\,(Au, z, t) * \mathcal{M}\,(z, z, t)\}.$$

$$\mathcal{M}\,(Au, z, kt) \ge \{1 * \mathcal{M}\,(Au, z, t) * 1 * \mathcal{M}\,(Au, z, t) * 1\}.$$

$$\mathcal{M}\,(Au, z, kt) \ge \{1 * \mathcal{M}\,(Au, z, t) * 1 * \mathcal{M}\,(Au, z, t) * 1\}.$$

which gives $\mathcal{M}\ (Au,z,kt) \geq \mathcal{M}\ (Au,z,t)$.

Therefore by the **Lemma 2. 2**, we have $Au = z$. Since $PQu = z$,

Thus $Au = PQu = z$, that is u is a coincidence point of A and PQ.

Similar to (3), we have

$$\begin{aligned}\mathcal{M}\ (y_{2n-1},Bv,kt) &= \mathcal{M}\ (Ax_{2n-2},Bv,kt)\\ &\geq \{\mathcal{M}\ (PQx_{2n-2},STv,t) * \mathcal{M}\ (Ax_{2n-2},PQx_{2n-2},t) * \mathcal{M}\ (Bv,STv,t)\\ &\quad * \mathcal{M}\ (Ax_{2n-2},STv,t) * \mathcal{M}\ (Bv,PQx_{2n-2},t)\}.\end{aligned}$$

Taking the limit as $n \to \infty$,

$$\mathcal{M}\ (z,Bv,kt) \geq \{\mathcal{M}\ (z,z,t) * \mathcal{M}\ (z,z,t) * \mathcal{M}\ (Bv,z,t) * \mathcal{M}\ (z,z,t) * \mathcal{M}\ (Bv,z,t)\}.$$

$$\mathcal{M}\ (z,Bv,kt) \geq \{1 * 1 * \mathcal{M}\ (Bv,z,t) * 1 * \mathcal{M}\ (Bv,z,t)\}.$$

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$$\mathcal{M}\ (z,Bv,kt) \geq \{1 * 1 * \mathcal{M}\ (Bv,z,t) * 1 * \mathcal{M}\ (Bv,z,t)\}.$$

which gives $\mathcal{M}\ (z,Bv,kt) \geq \mathcal{M}\ (Bv,z,t)$.

Therefore by the **Lemma 2. 2**, we have $Bv = z$. Since $STv = z$,

Thus $Bv = STv = z$, that is u is a coincidence point of B and ST.

Since the pair $\{A, PQ\}$ is the weakly compatible therefore A and PQ commute at their coincidence point that is $A(PQu) = PQ(Au)$ or $Az = PQz$. Similarly the pair $\{B, ST\}$ is the weakly compatible therefore B and ST commute at their coincidence point that is $B(STv) = ST(Bv)$ or $Bz = STz$.

Now we prove that $Az = z$, By (3), we have

$$\begin{aligned}&\mathcal{M}\ (Az,Bx_{2n-1},kt)\\ &\geq \{\mathcal{M}\ (PQz,STx_{2n-1},t) * \mathcal{M}\ (Az,PQz,t) * \mathcal{M}\ (Bx_{2n-1},STx_{2n-1},t)\\ &\quad * \mathcal{M}\ (Az,STx_{2n-1},t) * \mathcal{M}\ (Bx_{2n-1},PQz,t)\}.\end{aligned}$$

Taking the limit as $n \to \infty$, we have

$$\begin{aligned}\mathcal{M}\ (Az,z,kt) \geq \{&\mathcal{M}\ (PQz,z,t) * \mathcal{M}\ (Az,PQz,t) * \mathcal{M}\ (z,z,t) * \mathcal{M}\ (Az,z,t)\\ &* \mathcal{M}\ (z,PQz,t)\}.\end{aligned}$$

$$\mathcal{M}\ (Az,z,kt) \geq \{\mathcal{M}\ (z,z,t) * \mathcal{M}\ (Az,z,t) * \mathcal{M}\ (z,z,t) * \mathcal{M}\ (Az,z,t) * \mathcal{M}\ (z,z,t)\}.$$

$$\mathcal{M}\ (Az,z,kt) \geq \{1 * \mathcal{M}\ (Az,z,t) * 1 * \mathcal{M}\ (Az,z,t) * 1\}.$$

$$\mathcal{M}\ (Az,z,kt) \geq \{1 * \mathcal{M}\ (Az,z,t) * 1 * \mathcal{M}\ (Az,z,t) * 1\}.$$

which gives $\mathcal{M}\ (Az,z,kt) \geq \mathcal{M}\ (Az,z,t)$.

Therefore by **Lemma 2. 2**, we have $Az = z$. Since $PQu = Az$,

thus $Az = PQz = z$. Similar to (3), we have

$$\mathcal{M}\ (Ax_{2n-2}, Bz, kt) \geq \{\mathcal{M}\ (PQx_{2n-2}, STz, t) * \mathcal{M}\ (Ax_{2n-2}, PQx_{2n-2}, t) * \mathcal{M}\ (Bz, STz, t) * \mathcal{M}\ (Ax_{2n-2}, STz, t) * \mathcal{M}\ (Bz, PQx_{2n-2}, t)\}.$$

Taking the limit as $n \to \infty$, we have

$$\mathcal{M}\ (z, Bz, kt) \geq \{\mathcal{M}\ (z, z, t) * \mathcal{M}\ (z, z, t) * \mathcal{M}\ (Bz, z, t) * \mathcal{M}\ (z, z, t) * \mathcal{M}\ (Bz, z, t)\}.$$

$$\mathcal{M}\ (z, Bz, kt) \geq \{1 * 1 * \mathcal{M}\ (Bz, z, t) * 1 * \mathcal{M}\ (Bz, z, t)\}.$$

$$\mathcal{M}\ (z, Bz, kt) \geq \{1 * 1 * \mathcal{M}\ (Bz, z, t) * 1 * \mathcal{M}\ (Bz, z, t)\}.$$

$\mathcal{M}\ (z, Bz, kt) \geq \{1 * 1 * \mathcal{M}\ (Bz, z, t) * 1 * \mathcal{M}\ (Bz, z, t)\}.$

which gives $\mathcal{M}\ (z, Bz, kt) \geq \mathcal{M}\ (Bz, z, t)$.

Therefore by **Lemma 2.2**, we have $Bz = z$. Since $STz = Bz$,

thus $Bz = STz = z$.

For uniqueness, let w be another common fixed point of A, B, S, T, P and Q. By (3), we have

$$\mathcal{M}\ (z, w, kt) = \mathcal{M}\ (Az, Bw, kt)$$
$$\geq \{\mathcal{M}\ (PQz, STw, t) * \mathcal{M}\ (Az, PQz, t) * \mathcal{M}\ (Bw, STw, t) * \mathcal{M}\ (Az, STw, t) * \mathcal{M}\ (Bw, PQz, t)\}.$$

$$\mathcal{M}\ (z, w, kt) \geq \{\mathcal{M}\ (z, w, t) * \mathcal{M}\ (z, z, t) * \mathcal{M}\ (w, w, t) * \mathcal{M}\ (z, w, t) * \mathcal{M}\ (w, z, t)\}.$$

$$\mathcal{M}\ (z, w, kt) \geq \{\mathcal{M}\ (z, w, t) * \mathcal{M}\ (z, z, t) * \mathcal{M}\ (w, w, t) * \mathcal{M}\ (z, w, t) * \mathcal{M}\ (w, z, t)\}.$$

$$\mathcal{M}\ (z, w, kt) \geq \{\mathcal{M}\ (z, w, t) * 1 * 1 * \mathcal{M}\ (z, w, t) * \mathcal{M}\ (w, z, t)\}.$$

$$\mathcal{M}\ (z, w, kt) \geq \mathcal{M}\ (z, w, t).$$

From **Lemma 5.2**, $z = w$.

Therefore z is common fixed point of A, B, S, T, P and Q.

Theorem 5.2: Let $(X, \mathcal{M}, *)$ be a complete $\varepsilon -$ chainable fuzzy metric space and let A, B, S, T, P and Q be the self mappings of X, satisfying the following conditions:

(1) $A(X) \subset ST(X)$ and $B(X) \subset PQ(X)$;
(2) The pair (A, PQ) and (B, ST) are weakly compatible;
(3) There exists a constant $k \in (0, 1)$, such that for every $x, y \in X$ and $t > 0$,

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$$\mathcal{M}(Ax, By, kt) \geq \left\{ \mathcal{M}(PQx, STy, t) * \mathcal{M}(Ax, PQx, t) * \mathcal{M}(By, STy, t) * \mathcal{M}(Ax, STy, t) * \mathcal{M}(By, PQx, t) * \frac{\mathcal{M}(By, PQx, t)}{\mathcal{M}(PQx, STy, t) * \mathcal{M}(By, PQx, t)} * \frac{\mathcal{M}(PQx, STy, t) * \mathcal{M}(By, STy, t)}{\mathcal{M}(By, STy, t)} \right\}.$$

Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof: We can find a Cauchy sequence $\{y_n\}$ in X such that

$y_{2n-1} = STx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = PQx_{2n} = Bx_{2n-1}$ for $n = 1, 2, 3, \cdots$ From completeness, $y_n \to z$ for some $z \in X$, and so $\{Ax_{2n-2}\}, \{PQx_{2n}\}, \{Bx_{2n-1}\}$ and $\{STx_{2n-1}\}$ also converge to z.

Similarly we can show that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, hence there exists $z \in X$ such that $\{x_n\}$ converge to z.

Hence there exists $u, v \in X$ such that $PQu = z$ and $STv = z$ respectively.

By (3), we have

$$\mathcal{M}(Au, y_{2n}, kt) = \mathcal{M}(Au, Bx_{2n-1}, kt) \geq \left\{ \mathcal{M}(PQu, STx_{2n-1}, t) * \mathcal{M}(Au, PQu, t) * \mathcal{M}(Bx_{2n-1}, STx_{2n-1}, t) * \mathcal{M}(Au, STx_{2n-1}, t) * \mathcal{M}(Bx_{2n-1}, PQu, t) * \frac{\mathcal{M}(Bx_{2n-1}, PQu, t)}{\mathcal{M}(PQu, STx_{2n-1}, t) * \mathcal{M}(Bx_{2n-1}, PQu, t)} * \frac{\mathcal{M}(PQu, STx_{2n-1}, t) * \mathcal{M}(Bx_{2n-1}, STx_{2n-1}, t)}{\mathcal{M}(Bx_{2n-1}, STx_{2n-1}, t)} \right\}.$$

Taking the limit as $n \to \infty$,

$$\mathcal{M}(Au, z, kt) \geq \left\{ \mathcal{M}(z, z, t) * \mathcal{M}(Au, z, t) * \mathcal{M}(z, z, t) * \mathcal{M}(Au, z, t) * \mathcal{M}(z, z, t) * \frac{\mathcal{M}(z, z, t)}{\mathcal{M}(z, z, t) * \mathcal{M}(z, z, t)} * \frac{\mathcal{M}(z, z, t) * \mathcal{M}(z, z, t)}{\mathcal{M}(z, z, t)} \right\}.$$

$$\mathcal{M}(Au, z, kt) \geq \left\{ 1 * \mathcal{M}(Au, z, t) * 1 * \mathcal{M}(Au, z, t) * 1 * \frac{1}{1*1} * \frac{1*1}{1} \right\}.$$

$$\mathcal{M}(Au, z, kt) \geq \{1 * \mathcal{M}(Au, z, t) * 1 * \mathcal{M}(Au, z, t) * 1 * 1 * 1\}.$$

which gives $\mathcal{M}(Au, z, kt) \geq \mathcal{M}(Au, z, t)$.

Therefore by the **Lemma 5.2**, we have $Au = z$. Since $PQu = z$,

Thus $Au = PQu = z$, that is u is a coincidence point of A and PQ.

Similar to (3), we have

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$$\mathcal{M}\,(y_{2n-1}, Bv, kt) = \mathcal{M}\,(Ax_{2n-2}, Bv, kt)$$
$$\geq \Bigg\{\mathcal{M}\,(PQx_{2n-2}, STv, t) * \mathcal{M}\,(Ax_{2n-2}, PQx_{2n-2}, t) * \mathcal{M}\,(Bv, STv, t) * \mathcal{M}\,(Ax_{2n-2}, STv, t) * \mathcal{M}\,(Bv, PQx_{2n-2}, t) * \frac{\mathcal{M}\,(Bv, PQx_{2n-2}, t)}{\mathcal{M}\,(PQx_{2n-2}, STv, t) * \mathcal{M}\,(Bv, PQx_{2n-2}, t)} * \frac{\mathcal{M}\,(PQx_{2n-2}, STv, t) * \mathcal{M}\,(Bv, STv, t)}{\mathcal{M}\,(Bv, STv, t)}\Bigg\}.$$

Taking the limit as $n \to \infty$,

$$\mathcal{M}\,(z, Bv, kt) \geq \Bigg\{\mathcal{M}\,(z, z, t) * \mathcal{M}\,(z, z, t) * \mathcal{M}\,(Bv, z, t) * \mathcal{M}\,(z, z, t) * \mathcal{M}\,(Bv, z, t) * \frac{\mathcal{M}\,(Bv, z, t)}{\mathcal{M}\,(z, z, t) * \mathcal{M}\,(Bv, z, t)} * \frac{\mathcal{M}\,(z, z, t) * \mathcal{M}\,(Bv, z, t)}{\mathcal{M}\,(Bv, z, t)}\Bigg\}.$$

$$\mathcal{M}\,(z, Bv, kt) \geq \Bigg\{1 * 1 * \mathcal{M}\,(Bv, z, t) * 1 * \mathcal{M}\,(Bv, z, t) * \frac{\mathcal{M}\,(Bv, z, t)}{1 * \mathcal{M}\,(Bv, z, t)} * \frac{1 * \mathcal{M}\,(Bv, z, t)}{\mathcal{M}\,(Bv, z, t)}\Bigg\}.$$

$$\mathcal{M}\,(z, Bv, kt) \geq \Bigg\{1 * 1 * \mathcal{M}\,(Bv, z, t) * 1 * \mathcal{M}\,(Bv, z, t) * \frac{\mathcal{M}\,(Bv, z, t)}{\mathcal{M}\,(Bv, z, t)} * \frac{\mathcal{M}\,(Bv, z, t)}{\mathcal{M}\,(Bv, z, t)}\Bigg\}.$$

$$\mathcal{M}\,(z, Bv, kt) \geq \{1 * 1 * \mathcal{M}\,(Bv, z, t) * 1 * \mathcal{M}\,(Bv, z, t) * 1 * 1\}.$$

which gives $\mathcal{M}\,(z, Bv, kt) \geq \mathcal{M}\,(Bv, z, t)$.

Therefore by the **Lemma 5.2**, we have $Bv = z$. Since $STv = z$,

Thus $Bv = STv = z$, that is u is a coincidence point of B and ST.

Since the pair $\{A, PQ\}$ is the weakly compatible therefore A and PQ commute at their coincidence point that is $A(PQu) = PQ(Au)$ or $Az = PQz$.

Similarly the pair $\{B, ST\}$ is the weakly compatible therefore B and ST commute at their coincidence point that is $B(STv) = ST(Bv)$ or $Bz = STz$.

Now we prove that $Az = z$, By (3), we have

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$$\begin{aligned}\mathcal{M}\,(Az, Bx_{2n-1}, kt) &\geq \Big\{\mathcal{M}\,(PQz, STx_{2n-1}, t) * \mathcal{M}\,(Az, PQz, t) * \mathcal{M}\,(Bx_{2n-1}, STx_{2n-1}, t) \\ &* \mathcal{M}\,(Az, STx_{2n-1}, t) * \mathcal{M}\,(Bx_{2n-1}, PQz, t) \\ &* \frac{\mathcal{M}\,(Bx_{2n-1}, PQz, t)}{\mathcal{M}\,(PQz, STx_{2n-1}, t) * \mathcal{M}\,(Bx_{2n-1}, PQz, t)} \\ &* \frac{\mathcal{M}\,(PQz, STx_{2n-1}, t) * \mathcal{M}\,(Bx_{2n-1}, STx_{2n-1}, t)}{\mathcal{M}\,(Bx_{2n-1}, STx_{2n-1}, t)}\Big\}.\end{aligned}$$

Taking the limit as $n \to \infty$, we have

$$\begin{aligned}\mathcal{M}\,(Az, z, kt) \geq \Big\{&\mathcal{M}\,(PQz, z, t) * \mathcal{M}\,(Az, PQz, t) * \mathcal{M}\,(z, z, t) * \mathcal{M}\,(Az, z, t) \\ &* \mathcal{M}\,(z, PQz, t) * \frac{\mathcal{M}\,(z, PQz, t)}{\mathcal{M}\,(PQz, z, t) * \mathcal{M}\,(z, PQz, t)} \\ &* \frac{\mathcal{M}\,(PQz, z, t) * \mathcal{M}\,(z, z, t)}{\mathcal{M}\,(z, z, t)}\Big\}.\end{aligned}$$

$$\begin{aligned}\mathcal{M}\,(Az, z, kt) \geq \Big\{&\mathcal{M}\,(z, z, t) * \mathcal{M}\,(Az, z, t) * \mathcal{M}\,(z, z, t) * \mathcal{M}\,(Az, z, t) * \mathcal{M}\,(z, z, t) \\ &* \frac{\mathcal{M}\,(z, z, t)}{\mathcal{M}\,(z, z, t) * \mathcal{M}\,(z, z, t)} * \frac{\mathcal{M}\,(z, z, t) * \mathcal{M}\,(z, z, t)}{\mathcal{M}\,(z, z, t)}\Big\}.\end{aligned}$$

$$\mathcal{M}\,(Az, z, kt) \geq \Big\{1 * \mathcal{M}\,(Az, z, t) * 1 * \mathcal{M}\,(Az, z, t) * 1 * \frac{1}{1 * 1} * \frac{1 * 1}{1}\Big\}.$$

$$\mathcal{M}\,(Az, z, kt) \geq \{1 * \mathcal{M}\,(Az, z, t) * 1 * \mathcal{M}\,(Az, z, t) * 1 * 1 * 1\}.$$

which gives $\mathcal{M}\,(Az, z, kt) \geq \mathcal{M}\,(Az, z, t)$.

Therefore by **Lemma 5.2**, we have $Az = z$. Since $PQu = Az$,

Thus $Az = PQz = z$. Similar to (3), we have

$$\begin{aligned}\mathcal{M}\,(Ax_{2n-2}, Bz, kt) &\geq \Big\{\mathcal{M}\,(PQx_{2n-2}, STz, t) * \mathcal{M}\,(Ax_{2n-2}, PQx_{2n-2}, t) * \mathcal{M}\,(Bz, STz, t) \\ &* \mathcal{M}\,(Ax_{2n-2}, STz, t) * \mathcal{M}\,(Bz, PQx_{2n-2}, t) \\ &* \frac{\mathcal{M}\,(Bz, PQx_{2n-2}, t)}{\mathcal{M}\,(PQx_{2n-2}, STz, t) * \mathcal{M}\,(Bz, PQx_{2n-2}, t)} \\ &* \frac{\mathcal{M}\,(PQx_{2n-2}, STz, t) * \mathcal{M}\,(Bz, STz, t)}{\mathcal{M}\,(Bz, STz, t)}\Big\}.\end{aligned}$$

Taking the limit as $n \to \infty$, we have

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$$\mathcal{M}(z, Bz, kt) \geq \left\{\mathcal{M}(z, z, t) * \mathcal{M}(z, z, t) * \mathcal{M}(Bz, z, t) * \mathcal{M}(z, z, t) * \mathcal{M}(Bz, z, t) * \frac{\mathcal{M}(Bz, z, t)}{\mathcal{M}(z, z, t) * \mathcal{M}(Bz, z, t)} * \frac{\mathcal{M}(z, z, t) * \mathcal{M}(Bz, z, t)}{\mathcal{M}(Bz, z, t)}\right\}.$$

$$\mathcal{M}(z, Bz, kt) \geq \left\{1 * 1 * \mathcal{M}(Bz, z, t) * 1 * \mathcal{M}(Bz, z, t) * \frac{\mathcal{M}(Bz, z, t)}{1 * \mathcal{M}(Bz, z, t)} * \frac{1 * \mathcal{M}(Bz, z, t)}{\mathcal{M}(Bz, z, t)}\right\}.$$

$$\mathcal{M}(z, Bz, kt) \geq \left\{1 * 1 * \mathcal{M}(Bz, z, t) * 1 * \mathcal{M}(Bz, z, t) * \frac{\mathcal{M}(Bz, z, t)}{\mathcal{M}(Bz, z, t)} * \frac{\mathcal{M}(Bz, z, t)}{\mathcal{M}(Bz, z, t)}\right\}.$$

$$\mathcal{M}(z, Bz, kt) \geq \{1 * 1 * \mathcal{M}(Bz, z, t) * 1 * \mathcal{M}(Bz, z, t) * 1 * 1\}.$$

which gives $\mathcal{M}(z, Bz, kt) \geq \mathcal{M}(Bz, z, t)$.Therefore by **Lemma 5. 2**, we have $Bz = z$. Since $STz = Bz$, thus $Bz = STz = z$.

For uniqueness, let w be another common fixed point of A, B, S, T, P and Q. By (3), we have $\mathcal{M}(z, w, kt) = \mathcal{M}(Az, Bw, kt)$

$$\geq \left\{\mathcal{M}(PQz, STw, t) * \mathcal{M}(Az, PQz, t) * \mathcal{M}(Bw, STw, t) * \mathcal{M}(Az, STw, t) * \mathcal{M}(Bw, PQz, t) * \frac{\mathcal{M}(Bw, PQz, t)}{\mathcal{M}(PQz, STw, t) * \mathcal{M}(Bw, PQz, t)} * \frac{\mathcal{M}(PQz, STw, t) * \mathcal{M}(Bw, STw, t)}{\mathcal{M}(Bw, STw, t)}\right\}.$$

$$\mathcal{M}(z, w, kt) \geq \left\{\mathcal{M}(z, w, t) * \mathcal{M}(z, z, t) * \mathcal{M}(w, w, t) * \mathcal{M}(z, w, t) * \mathcal{M}(w, z, t) * \frac{\mathcal{M}(w, z, t)}{\mathcal{M}(z, w, t) * \mathcal{M}(w, z, t)} * \frac{\mathcal{M}(z, w, t) * \mathcal{M}(w, w, t)}{\mathcal{M}(w, w, t)}\right\}.$$

$$\mathcal{M}(z, w, kt) \geq \left\{\mathcal{M}(z, w, t) * 1 * 1 * \mathcal{M}(z, w, t) * \mathcal{M}(w, z, t) * 1 * \frac{\mathcal{M}(z, w, t) * 1}{1}\right\}.$$

$$\mathcal{M}(z, w, kt) \geq \{\mathcal{M}(z, w, t) * 1 * 1 * \mathcal{M}(z, w, t) * \mathcal{M}(w, z, t) * 1 * \mathcal{M}(z, w, t)\}.$$

Which gives $\mathcal{M}(z, w, kt) \geq \mathcal{M}(z, w, t)$. From **Lemma 5. 2**, $z = w$.

Therefore z is common fixed point of A, B, S, T, P and Q.

4 Conclusion

In this paper, we prove a fixed point theorem for six weakly compatible mappings within a complete, chainable fuzzy metric space, without requiring continuity of the mappings. The results presented offer broader applicability and hold significant value for further research in this area.

5 References

[1]. Ali Syed Shahnawaz, Jain Jainendra, Sanodia P.L. & Jain Shilpi (2017). **An Extended Result on Fixed Point Theorem in $\varepsilon -$ Chainable Fuzzy Metric Space.** International Journal of Current Engineering and Technology, vol. 7(2), April 2017, pp. 374 – 377.

[2]. Cho Y. J. (1997). **Fixed Point in Fuzzy Metric Space.** Journal of Fuzzy Mathematics, vol. 4, pp. 949 – 962.

[3]. George A. and Veeramani P. (1994) **On Some Results in Fuzzy Metric Spaces.** Fuzzy Sets and Systems, vol. 64, pp. 395 – 399.

[4]. Grabiec M. (1988). **Fixed Points in Fuzzy Metric Spaces.** Fuzzy Sets and Systems, vol. 27, pp. 385 – 389.

[5]. Jungck G. (1986). **Compatible Mappings and Common Fixed Points.** International Journal of Mathematics and Mathematical Sciences, vol. 9 (4), pp. 771 – 779.

[6]. Jungck G. and Rhoades B. E. (2006). **Fixed Point Theorems for Occasionally Weakly Compatible Mappings.** Fixed Point Theory, vol. 7 (2), pp. 287 – 296.

[7]. Jungck G. and Rhoades B.E. (1998). **Fixed Points for Set Valued Functions without Continuity,** Indian Journal of Pure and Applied Mathematics, vol. 29 (3), pp. 227 – 238.

[8]. Kramosil I. and Michalek J. (1975) **Fuzzy Metric and Statistical Metric Spaces.** Kybernetica, vol. 11, pp. 326 – 334.

[9]. Sessa S. (1982). **On Weak Commutativity Condition of Mapping in Fixed Point Consideration**, Publ. Inst. Math. (Beograd) N.S., vol. 32(46), pp. 149 – 153.

[10]. Sharma S. and Deshpande B. (2009). **Common Fixed Point Theorems for Finite Number of Mappings without Continuity and Compatibility on intuitionistic Fuzzy Metric Spaces.** Chaos, Solitons and Fractals, vol. 40, pp. 2242 – 2256.

[11]. Sharma S. and Deshpande B. (2010). **Common Fixed Point Theorems for Finite Number of Mappings without Continuity and Compatibility on Fuzzy Metric Spaces.** Fuzzy Sets and Systems, vol. 24(2), pp. 73 – 83.

[12]. Singh B. and Chauhan M. S. (2000). **Common Fixed Points of Compatible Maps in Fuzzy Metric Spaces**, Fuzzy Sets and System, Vol. 115, pp. 471 – 475.

[13]. Singh B. and Jain S. (2005). **Semi - Compatibility and Fixed Point Theorems in Fuzzy Metric Spaces using Implicit Relation.** International Journal of Mathematics and Mathematical Sciences, vol. 16, pp. 2617 – 2629.

[14]. Vasuki R. (1999). **Common Fixed Points for R – weakly Commuting Maps in Fuzzy Metric Spaces.** Indian Journal of Pure and Applied Mathematics, vol. 30 (4), pp. 419 – 423.

[15]. Zadeh L. (1965). **Fuzzy Sets**. Inform and control, vol. 8, pp. 338 – 353.