



Rethinking the Pedagogical Sequence of Trigonometric Formulae in Secondary Education : A Preliminary Assessment of an Alternative Teaching Approach in a Hong Kong Secondary School

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Abstract

The teaching of trigonometry occupies an important position in secondary mathematics education because it serves as a bridge between geometry, algebra, and advanced mathematical modelling. Despite its importance, many students encounter significant difficulties when solving non-right-angled triangles using the sine and cosine formulae. In Hong Kong secondary schools, the conventional textbook sequence typically introduces the sine formula through angle-side-angle cases, proceeds to the ambiguous side-side-angle case, and subsequently introduces the cosine formula through side-angle-side and side-side-side situations. While this sequence reflects mathematical logic and curriculum traditions, its pedagogical effectiveness has received relatively little scrutiny. This conceptual study re-examines the sequencing of trigonometric formulae from the perspective of classroom teaching and learning. Drawing upon reflective practice developed through secondary mathematics teaching in Hong Kong, the study argues that the conventional ordering may impose unnecessary cognitive demands on learners, particularly when the ambiguous case of the sine formula is introduced prematurely. An alternative pedagogical approach is proposed whereby students first encounter area formulae, followed by the sine rule as a tool for determining unknown lengths, while the cosine rule is employed strategically to reduce ambiguity and support exact algebraic solutions.

The manuscript analyses the mathematical structure of solving triangles, common student misconceptions, examination errors, and the interaction between conceptual understanding and procedural fluency. Relevant theoretical perspectives, including cognitive load theory, constructivism, variation theory, and APOS theory, are employed to frame the discussion. Implications for curriculum design, textbook development, teacher education, assessment practices, and future research are also explored.

Rather than presenting a large-scale empirical investigation, this study contributes a theoretically informed pedagogical framework intended to stimulate professional dialogue concerning the teaching of trigonometry in secondary schools. The analysis suggests that curriculum sequencing deserves greater attention and that alternative instructional pathways may better support students' conceptual understanding and problem-solving competence.



Keywords

Trigonometry; Sine Rule; Cosine Rule; Mathematics Education; Curriculum Design; Pedagogical Sequence; Cognitive Load Theory; Hong Kong Secondary Education; Solving Triangles; Mathematics Teaching

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Chapter 1

Introduction and Background

1.1 The Importance of Trigonometry in Secondary Education

Trigonometry has long occupied a central position in school mathematics. Beyond its computational applications, it serves as a critical meeting point between geometric reasoning and algebraic manipulation. Students are expected to transition from concrete geometric relationships to abstract symbolic representations, making trigonometry one of the first major areas in secondary mathematics where multiple representations interact extensively.

The study of trigonometry provides learners with tools for modelling physical phenomena, measuring inaccessible distances, analysing periodic behaviour, and solving geometric problems. Applications can be observed in architecture, engineering, navigation, surveying, astronomy, computer graphics, physics, and data science. Consequently, the effectiveness with which trigonometry is taught may influence students' readiness for advanced studies in mathematics and related disciplines.

For many students, however, trigonometry becomes a source of frustration rather than empowerment. Difficulties frequently arise from remembering formulas, selecting appropriate methods, interpreting diagrams, and understanding the meaning of mathematical relationships. These challenges become particularly evident when students encounter non-right-angled triangles.

1.2 Trigonometry as a Gateway Topic

In the secondary curriculum, right-angled trigonometry is typically introduced first. Students learn the sine, cosine, and tangent ratios within the context of right triangles. Because the relationships are closely linked to geometric diagrams and direct measurements, many learners are able to develop a procedural understanding of basic applications.

The transition to solving non-right-angled triangles represents a significant conceptual leap. Students must no longer rely on simple right-triangle definitions. Instead, they are introduced to more sophisticated relationships, including:

- the sine formula;
- the cosine formula;
- area formulae involving trigonometric ratios;
- ambiguous cases;
- exact and approximate solutions;
- multiple solution scenarios.

This transition requires students to coordinate several mathematical ideas simultaneously:

1. geometric structure;
2. algebraic manipulation;
3. trigonometric functions;
4. logical reasoning;
5. calculator operations.

Consequently, errors increase dramatically during this stage of learning.



1.3 The Original Paper

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Reflections on teaching sine and cosine formulae in solving triangles

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Introduction

In the topic “More about trigonometry”, most mathematics textbooks in Hong Kong will present in this order: the use of the sine formula in the A.S.A. (or A.A.S.) triangle, the use of the sine formula in the S.S.A. triangle (the ambiguous case), the use of the cosine formula in the S.A.S. triangle, and then the use of the cosine formula in the S.S.S. triangle. After that, a lot of drillings and practices will be put in the exercises on solving triangles (i.e. finding the lengths of 3 sides and the sizes of 3 internal angles of a triangle) and applications to 2-dimensional problems.

However, is it the best approach to introduce the topic? In this article, the writer will share some of the thoughts in introducing the concepts of these two formulae besides using the “textbook approach”.

The sine formula

The formula for the area of a S.A.S. triangle $\frac{1}{2}ab \sin C$ can be introduced before the introduction of sine formula. The proof of this formula requires only junior form trigonometry, and by dividing abc from this formula, the sine formula can be proved.

For the A.A.S. triangle, it is enough to emphasize the use of sine formula to find the length of a side of triangle, and that it has one solution only.

For the S.S.A. triangle, most textbooks try to compare the length of a side of the triangle (without loss of generality, the length of “ a ” opposite to the angle “ A ”) with the height of the triangle (the perpendicular distance from the vertex C to its opposite side) under different cases of the size of A , i.e. acute, right or

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obtuse. This approach is quite difficult for most average students because it creates a lot of noise to them. So, the writer will just emphasize that the sine formula is applied to find the size of the angle. (And later on, the writer will discuss the fact that the cosine formula can be applied as an alternative method to find the length of the third side.) For using the sine formula, there will be no solution when $\sin \theta > 1$ or angle sum of triangle $> 180^\circ$. On the other hand, there will be 1 solution when only one angle with angle sum of triangle $< 180^\circ$. 2 solutions when angle in Quad I $>$ given angle.

If there is time, the writer will discuss with students the condition for the “only one solution” for the S.S.A. triangle.

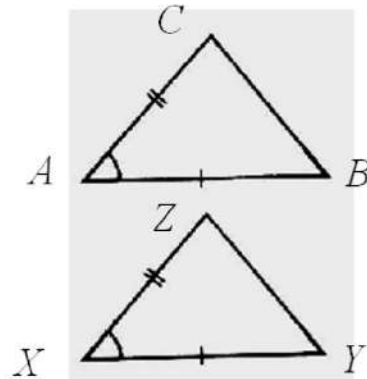


Figure 1

In the above figure, let $\angle BAC$ (or $\angle YXZ$) be the included angle (“A”).

S_2S_1A is not a condition for congruence (that is, has more than 1 solution), where S_2 , the side opposite to “A”, is shorter than S_1 , one of the “arms” of the included angle, i.e. $A < 90^\circ$.

Counter example: $AC = 6$ cm, $\angle BAC = 30^\circ$ and $BC = 4$ cm. Then, by sine formula, $\sin \angle ABC = 0.75$ and there will be two solutions ($\angle ABC \approx 48.5904^\circ$ or 131.4096°).

The cosine formula

For the S.A.S. triangle, the cosine formula is used to find out the third side first. Then we can find the second angle by either sine formula or the cosine

formula. The writer thinks that applying cosine formula twice is too complicated for students (especially when they have no programme input in their scientific calculator). Therefore, the writer will emphasize that even if sine formula is used, for the second angle and the third angle, we always find the smaller one to avoid ambiguity. That is, for a S.A.S. triangle, find out the third side first. Then,

- if the largest angle is given, find out the smallest angle.
- if the medium angle is given, find out the smallest angle.
- if the smallest angle is given, find out the medium angle.

Of course, after finishing this section, the writer will discuss with students which formula is more convenient and let them choose their own way. (The ambiguity verses the unique solution? The one without the need for a calculator programme verses the one requires two executions of one calculator programme - the programme for finding the side and the programme for finding the angle?)

For the S.S.S. triangle, most textbooks will emphasize the finding of the largest angle first by cosine formula (that is, the angle opposite to the longest side) to avoid ambiguity (because then the second angle and the third angle will not have two solutions even when sine formula is used). But then, they just ask the students to find the second angle by cosine formula again, and then the third angle by angle sum of triangle. Again, the writer thinks that applying cosine formula twice is too complicated for students (especially when they have no programme input in their scientific calculator). Therefore, again the writer will emphasize that even if sine formula is used, for the second angle and the third angle, we always find the smaller one to avoid ambiguity. That is, for a S.S.S. triangle, always find out the smallest angle first by cosine formula. (For cosine formula, there will be no solution when cosine value < -1 or cosine value > 1 .) Then, find out the medium angle by sine formula. Finally, use angle sum of triangle to find the remaining angle (the largest angle).

The writer will also introduce the use of cosine formula for the S.S.A. triangle. A quadratic equation in the length of the third side will be formed and the number of roots will vary according to the discriminant of the quadratic

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equation formed. The aim of introducing this method is that when a question asks for the exact value of the third length, the cosine formula will give a root in surd form by using the quadratic equation formula. For abler students, the writer will also discuss the number of roots, i.e. for cosine formula, no solution when the discriminant < 0 or the two roots are both negative.

For cosine formula, 1 solution when product of roots < 0 (i.e. one of the roots is positive and the other root is negative).

For abler students, the writer will ask them if there is a need for creating a “tangent formula”. There is no urgent need for setting up such a formula because the existing sine formula and cosine formula are already enough for solving different types of (congruent) triangles we learnt in junior form. (Actually, there is such a “tangent formula” for S.A.S. or A.S.A. or A.A.S.

triangles. The law is $\frac{a-b}{a+b} = \frac{\tan \frac{A-B}{2}}{\tan \frac{A+B}{2}}$.)

Here is a summary of what I intended to discuss in the class.

Given	3 sides (S.S.S.)	2 sides and 1 angle		1 side and 2 angles (A.S.A. or A.A.S.)
		included angle (S.A.S.)	Non-included opposite angle (S.S.A.)	
Number of solutions	0 or 1	1	0 or 1 or 2	1
	(No solution when $\cos \theta < -1$ or $\cos \theta > 1$)		(For sine formula, no solution when $\sin \theta > 1$ or angle sum of triangle $> 180^\circ$. For cosine formula, no solution when discriminant < 0 or the two roots are negative.) 1 solution when only one angle with angle sum $<$ 180° or product of roots < 0 .	
Method of solutions	cosine formula		sine formula	sine formula to find the <i>length</i>
	to find the smallest <i>angle</i> , then use sine formula (easier but need to check) or cosine formula (better but a bit complicated) for the S.S.S.A. triangle	to find the <i>length</i>	(if exact value of the third side are NOT required) to find the <i>angle</i> or cosine formula (if exact value of the third side are required) to find the <i>length</i>	

Figure 2

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I would like to end by discussing the common errors in two questions appeared in the final mathematics exam paper in my school.

Question 1: In the figure, PRS is a straight line, $PQ = 10$ cm, $QR = QS = 7$ cm and $\angle QPR = 35^\circ$. Find the size of $\angle PQR$.

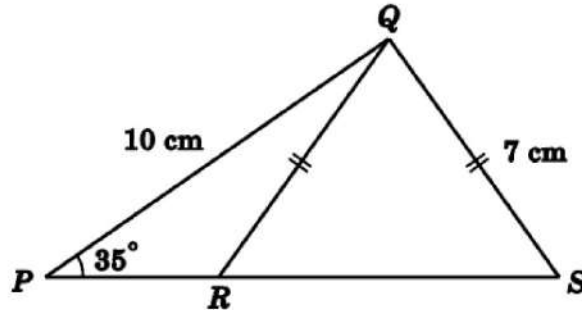


Figure 3

Here is the marking scheme.

In $\triangle PQR$, by the sine formula, $\frac{7 \text{ cm}}{\sin 35^\circ} = \frac{10 \text{ cm}}{\sin \angle PRQ}$	1M
$\angle PRQ = 55.0243^\circ$, corr. to 6 sig. fig.	
or $\angle PRQ = 180^\circ - 55.0243^\circ = 124.976^\circ$	1A
$\therefore$ Given PSR is a st. line and $\triangle QRS$ is isos., so $\angle PRQ$ is an obtuse angle.	
$\therefore \angle PRQ = 124.976^\circ$	
$\angle PQR = 180^\circ - 35^\circ - 124.976^\circ$ $\angle$ sum of $\triangle$	
$= \boxed{20.0243} = \underline{\underline{20.0^\circ}}$, corr. to 3 sig. fig. (NOT accept 89.9757°)	1A

A lot of students are having difficulty in finding the size of $\angle PRQ$. They just give 55.0243° as their answer, without considering the angle in the second quadrant.

Question 2: In the figure, ship A and ship B are 15 km apart. The bearings of ship A and ship B from the lighthouse L are $S50^\circ E$ and $N60^\circ E$ respectively. The bearing of ship B from ship A is $N25^\circ W$.

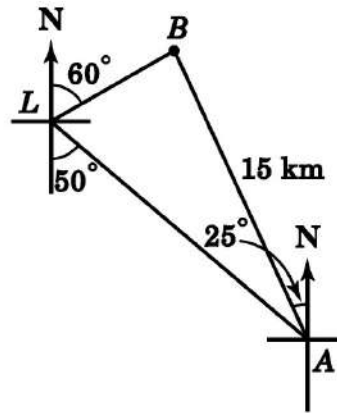


Figure 4

- (a) Find the distance between ship B and lighthouse L .
- (b) John claims that the distance between ship A and lighthouse L is greater than that between ship B and lighthouse L . Do you agree with John? Explain your answer.

Here is the “initial” marking scheme.

(a)	$60^\circ + \angle BLA + 50^\circ = 180^\circ$	
	$\angle BLA = 70^\circ$	1A
	$25^\circ + \angle BAL = 50^\circ$	
	$\angle BAL = 25^\circ$	or 1A (either one)
	In $\triangle BAL$, by the sine formula, $\frac{BL}{\sin \angle BAL} = \frac{AB}{\sin \angle BLA}$	1M
	$BL = \frac{15 \sin 25^\circ}{\sin 70^\circ} \text{ km} = 6.746114406 \text{ km}$	
	$\therefore$ The distance between ship B and lighthouse L is <u>6.75 km</u> .	1A
(b)	In $\triangle BAL$, $\angle LBA + 25^\circ + 70^\circ = 180^\circ$	
	$\angle LBA = 85^\circ$	1A
	By the sine formula, $\frac{AL}{\sin \angle LBA} = \frac{AB}{\sin \angle BLA}$	1M
	$AL = \frac{15 \sin 85^\circ}{\sin 70^\circ} \text{ km} = 15.9019 \text{ km, corr. to 6 sig. fig.}$	
	which is greater than BL .	
	$\therefore$ John's claim is <u>agreed</u> .	1A must with explanation

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I say it is an “initial” marking scheme because after marking nearly 190 scripts of exam papers, only one girl, who is not good in the past mathematics examinations, can explain by using the fact that “larger angle, larger side”. Here is what she writes.

$$\angle LBA = 180^\circ - 25^\circ - 70^\circ = 85^\circ > 25^\circ = \angle BAL, \text{ so } AL > BL.$$

And hence John's claim is agreed.

Note that the longest side is opposite to the largest angle, and vice-versa. Therefore, when answering part (b), actually there is no need to calculate the length of AL by sine formula. I have included her solution in the finalized marking scheme, so that other students can think if there can be more than one method of solution to the same problem. Sine formula and cosine formula sometimes may not be the most powerful method!

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1.4 Teaching Challenges in Solving Triangles

The original reflection paper arose from practical classroom experiences teaching "More About Trigonometry" in a Hong Kong secondary school. The paper raised an important pedagogical question: although textbooks present the sine and cosine formulae in a widely accepted order, is that order necessarily optimal from the perspective of student learning?

Most textbooks follow a sequence that begins with:

1. A.A.S./A.S.A. triangles using the sine formula.
2. S.S.A triangles using the sine formula.
3. S.A.S. triangles using the cosine formula.
4. S.S.S. triangles using the cosine formula.

This ordering is mathematically coherent. However, from a classroom perspective, several difficulties arise.

First, the ambiguous S.S.A case appears relatively early in instruction. Students encounter situations where two different triangles satisfy given conditions, one triangle satisfies the conditions, or no triangle satisfies the conditions. Such complexity often overwhelms learners who have only recently become familiar with the sine rule itself.

Second, students frequently struggle to determine whether the angle produced by the inverse sine function represents the required solution. Many fail to recognize that the supplementary angle may also satisfy the equation. As shown in classroom assessment examples, students commonly select the acute angle generated directly from the calculator without considering the geometric context.

Third, solving S.A.S. and S.S.S. triangles often requires repeated application of the cosine formula. While mathematically correct, this can become computationally burdensome for students and may obscure conceptual understanding. The original paper noted that applying the cosine formula repeatedly can be particularly challenging for students who lack calculator programming capabilities.

These observations suggest that the traditional sequence may place excessive emphasis on procedural complexity rather than conceptual progression.

1.5 The Hong Kong Mathematics Curriculum Context

Hong Kong's mathematics curriculum has historically emphasized strong procedural competence and examination performance. Students are expected to develop proficiency in algebraic manipulation, precise computation, and logical reasoning. These strengths have contributed to high levels of mathematical achievement internationally.

Within this context, trigonometry serves as an important examination topic. Questions often require students to:

- determine unknown sides;
- calculate unknown angles;
- interpret geometrical diagrams;
- apply mathematical reasoning;
- justify conclusions.

As a result, teachers frequently face pressure to prioritize efficient examination techniques alongside conceptual understanding.



The experiences documented in the original reflection illustrate a tension familiar to many mathematics teachers. Should instruction prioritize mathematically elegant derivations and textbook conventions, or should it prioritize approaches that reduce cognitive burden and improve accessibility for learners?

1.6 Why Rethink Pedagogical Sequences?

Curriculum content and curriculum sequence are not identical concepts.

Educational systems often devote considerable attention to determining what should be taught while dedicating less attention to the order in which concepts should be introduced. Yet psychological research suggests that sequencing plays a crucial role in learning outcomes.

A mathematical topic can often be introduced through multiple pathways. For example:

- historical sequence;
- logical sequence;
- textbook sequence;
- examination-oriented sequence;
- cognitively optimized sequence.

These different sequences may produce substantially different learning experiences.

The central argument of this manuscript is that solving triangles should be viewed not merely as a collection of formulas but as an interconnected network of mathematical relationships. When students perceive the formulas as isolated procedures, they tend to memorize rather than understand. When relationships between formulas are emphasized, students are better positioned to reason flexibly and select methods strategically.

The original article demonstrated this philosophy by connecting:

- area formulas,
- sine formulas,
- cosine formulas,
- exact solutions,
- ambiguity,
- geometric reasoning.

Such connections form the basis of the alternative pedagogical framework developed throughout this manuscript.

1.7 Purpose of the Study

The purpose of this conceptual study is fourfold.

First, it seeks to critically examine the conventional sequence used in teaching trigonometric formulae.

Second, it aims to identify sources of student difficulty associated with existing instructional practices.

Third, it proposes an alternative pedagogical sequence grounded in classroom experience and educational theory.

Fourth, it explores implications for curriculum development, textbook design, assessment, and teacher professional learning.

1.8 Research Questions

This manuscript is guided by the following questions:

RQ1: What assumptions underlie the conventional sequence used for teaching sine and cosine formulae?

RQ2: What conceptual and procedural difficulties arise from this sequence?

RQ3: How can cognitive and pedagogical theories be used to evaluate alternative instructional pathways?

RQ4: What advantages and limitations emerge from an alternative sequence emphasizing conceptual coherence and reduced ambiguity?

RQ5: How might alternative sequencing influence curriculum and assessment practices in Hong Kong secondary education?

1.9 Significance of the Study

The significance of this study lies in its focus on a frequently overlooked aspect of mathematics education: sequencing.

Most discussions concerning trigonometry focus on:

- curriculum coverage;
- examination difficulty;
- technological tools;
- student achievement.

Far less attention is directed towards the instructional order through which concepts are encountered.

The present manuscript argues that sequencing decisions may influence:

- conceptual understanding;
- procedural fluency;
- error rates;
- confidence;
- transferability of knowledge.

The analysis therefore contributes to curriculum discourse at classroom, school, and policy levels.

1.10 Organization of the Manuscript

Following this introductory chapter, Chapter 2 examines the historical development of trigonometry and its educational treatment across different periods and traditions.

Chapter 3 reviews literature concerning student learning difficulties, misconceptions, procedural fluency, and curriculum sequencing.

Chapter 4 establishes the theoretical framework underpinning the analysis.

Chapter 5 critically examines the conventional instructional sequence currently prevalent in Hong Kong textbooks.

Chapter 6 presents a detailed alternative pedagogical framework inspired by reflective practice.

Chapter 7 illustrates classroom implementation and preliminary observations.

Chapter 8 analyses examination errors and student reasoning processes.

Chapter 9 considers implications for teachers, textbook writers, curriculum developers, and policymakers.

Finally, Chapter 10 summarizes the study and proposes future research directions.

Chapter 2

Historical Development of Trigonometry Education

2.1 Introduction

The teaching of trigonometry in contemporary secondary schools reflects the accumulated influence of more than two millennia of mathematical development. Although students often encounter trigonometric formulae as established rules to be memorized and applied, the subject itself emerged gradually through practical human needs involving astronomy, navigation, surveying, architecture, and geographic measurement. Understanding this historical evolution provides important insights into how trigonometry has come to occupy its present position within school mathematics and why particular instructional sequences have become dominant.

From a pedagogical perspective, historical analysis demonstrates that mathematical concepts were not originally developed in the order commonly presented in modern textbooks. Rather, trigonometric knowledge evolved through a complex interaction of practical problem-solving, theoretical abstraction, and technological advancement. Consequently, examining the historical development of trigonometry offers an opportunity to reconsider whether current instructional sequences are pedagogically optimal or merely products of historical convention.

This chapter traces the evolution of trigonometry from its early origins in ancient civilizations to its modern treatment in secondary mathematics curricula. Particular attention is given to the development of triangle-solving techniques and the eventual emergence of the sine and cosine formulae that are central to contemporary instruction.

2.2 Early Origins of Trigonometric Thinking

The roots of trigonometry can be traced to ancient civilizations long before the establishment of trigonometry as a formal mathematical discipline. Early societies required methods for measuring distances, constructing buildings, predicting celestial events, and determining geographic positions.

2.2.1 Ancient Egyptian Contributions

The ancient Egyptians demonstrated considerable sophistication in geometry and measurement. Their construction of pyramids required accurate knowledge of slopes, angles, and proportional relationships. Although explicit trigonometric functions were not yet developed, mathematical ideas underlying trigonometric reasoning were already evident.

For example, Egyptian surveyors used ratios of vertical rise to horizontal run when constructing monuments. The concept known as the "seqed" described the inclination of pyramid faces and served a function remarkably similar to the modern tangent ratio.

These measurement techniques reveal an important historical lesson. Practical geometric problems preceded formal trigonometric theory. Human beings sought solutions to real-world challenges long before they developed abstract mathematical frameworks.

From an educational perspective, this observation suggests that trigonometric concepts may be understood more naturally when introduced through meaningful geometric situations rather than through formula memorization alone.

2.2.2 Babylonian Mathematics

Babylonian civilization made substantial contributions to numerical computation and angular measurement. Their sexagesimal (base-60) number system continues to influence modern mathematics through the division of circles into 360 degrees and subdivisions of degrees into minutes and seconds.

Babylonian astronomers compiled extensive observational records and developed computational techniques for predicting celestial motions. Although these techniques did not constitute trigonometry in the modern sense, they laid essential foundations for future mathematical developments.

The Babylonian emphasis on numerical tables would later influence the development of trigonometric tables, an instructional feature that remained important until the widespread availability of electronic calculators.

2.3 Greek Foundations of Trigonometry

The formal development of trigonometry is often associated with ancient Greek mathematics, particularly the work of astronomers seeking to model the motion of celestial bodies.

2.3.1 Geometry as the Language of Astronomy

Ancient Greek scholars viewed geometry as the principal framework for understanding the physical universe. Astronomical observations required accurate predictions of planetary positions, lunar eclipses, and seasonal cycles. These practical demands stimulated the development of increasingly sophisticated geometric methods.

Rather than using sine and cosine functions, Greek mathematicians focused on chords of circles. A chord is a line segment joining two points on a circle. Problems involving arcs and chords eventually evolved into relationships that closely resemble modern trigonometric functions.

This historical fact has significant educational implications. Trigonometry did not emerge independently of geometry; instead, it arose from geometric reasoning. Modern instructional approaches that separate trigonometry from geometry may therefore obscure important conceptual relationships.

2.3.2 Hipparchus and the First Trigonometric Tables

Hipparchus (c.190 BCE-c.120 BCE) is frequently regarded as the "Father of Trigonometry." His astronomical investigations led to the construction of chord tables that enabled more precise calculations involving circles and angles.

The significance of Hipparchus lies not merely in computational innovation but in the idea of creating systematic numerical relationships between geometric quantities. This represented an important shift from isolated geometric constructions toward generalized mathematical procedures.

Many educational historians regard the creation of trigonometric tables as one of the earliest examples of transforming geometric knowledge into computational algorithms.

2.3.3 Ptolemy and Mathematical Consolidation

The work of Claudius Ptolemy (c.100 CE-c.170 CE) further advanced trigonometric methods. In his influential work *Almagest*, Ptolemy presented extensive chord tables and developed relationships among sides and angles in cyclic quadrilaterals.

Ptolemy's theorem and associated techniques provided powerful tools for astronomical calculations. More importantly, they illustrated a recurring characteristic of trigonometric development: the close interaction between theoretical mathematics and practical application.

This interaction remains important in secondary education. Students often encounter trigonometric formulae as isolated mathematical objects despite their origins in solving significant real-world measurement problems.

2.4 Indian Contributions and the Birth of the Sine Function

The transition from chord-based methods to modern trigonometric functions occurred primarily through the work of Indian mathematicians.

2.4.1 Emergence of the Sine Concept

Indian scholars introduced the concept of half-chords, which eventually evolved into the sine function. This development greatly simplified many astronomical calculations.

The Sanskrit term *jya* referred to a half-chord. Through successive translations into Arabic and later Latin, this concept ultimately became known as the sine function.

The educational significance of this development cannot be overstated. The shift from chords to sines illustrates how mathematical ideas evolve toward greater efficiency and simplicity. Concepts that appear natural and obvious today are often the result of centuries of refinement.

2.4.2 Aryabhata's Influence

Aryabhata (476-550 CE) produced sophisticated trigonometric tables and introduced methods that closely resemble modern trigonometric calculations.

His work demonstrated increasing abstraction in mathematical thinking. Rather than viewing each geometric problem individually, trigonometric relationships were recognized as general principles applicable across broad classes of situations.

This movement toward generalization mirrors one of the central objectives of mathematics education. Students are expected not only to solve individual problems but also to recognize underlying structures and relationships.

2.5 Islamic Contributions to Trigonometry

Between the eighth and fifteenth centuries, Islamic scholars played a crucial role in preserving, extending, and systematizing trigonometric knowledge.

2.5.1 Independent Development of Trigonometry

Islamic mathematicians transformed trigonometry into an independent mathematical discipline distinct from astronomy.

During this period:

- sine functions were systematically studied;
- cosine functions were developed;
- tangent functions were formalized;
- spherical trigonometry expanded rapidly;
- trigonometric identities emerged.

The transition from practical computation to theoretical organization represented a major milestone in mathematical development.

2.5.2 Development of Triangle-Solving Methods

Many techniques used in modern classrooms can trace their intellectual origins to this period.

Islamic scholars developed procedures for solving triangles, establishing relationships among sides and angles, and deriving identities that formed the basis for later trigonometric formulae.

Importantly, these developments did not occur in the order typically used in contemporary textbooks. Mathematical ideas evolved organically in response to specific practical needs.

This observation supports the broader argument of this manuscript: historical development and pedagogical effectiveness need not coincide.

2.6 The Emergence of Modern Trigonometry

2.6.1 Renaissance Mathematics

The European Renaissance witnessed renewed interest in mathematical sciences. Advances in navigation, cartography, engineering, and astronomy increased demand for powerful computational techniques.

The works of Regiomontanus and subsequent European mathematicians helped establish trigonometry as a distinct branch of mathematics.

The publication of dedicated trigonometry textbooks signaled an important transition. Knowledge that had previously been embedded within astronomy became a standalone subject suitable for systematic instruction.

2.6.2 Logarithms and Computational Efficiency

The invention of logarithms during the seventeenth century dramatically transformed trigonometric calculations. Before electronic calculators, solving triangles often required lengthy numerical procedures involving table lookups. Students spent substantial instructional time mastering computational techniques that are now performed instantly by calculators.

This historical perspective invites reflection on modern instructional priorities. If technology has removed much of the computational burden, greater emphasis may be placed on conceptual understanding and strategic method selection.

The concerns raised in the original reflection regarding repeated use of the cosine formula and calculator dependence can therefore be understood within a broader historical context. The balance between computational efficiency and conceptual understanding has long been a challenge in mathematics education.

2.7 Trigonometry in School Mathematics

2.7.1 Nineteenth-Century Curriculum Development

During the nineteenth century, many educational systems incorporated trigonometry into secondary schooling.

The dominant instructional model emphasized:

- formal proof;
- derivation of identities;

- manipulation of formulae;
- mastery of tables;
- geometric applications.

This tradition influenced generations of textbook writers and continues to shape instructional practices today.

2.7.2 Twentieth-Century Reforms

Educational reforms during the twentieth century sought to increase conceptual understanding and problem-solving competence.

Several important changes occurred:

1. Reduced dependence on logarithmic tables.
2. Increased use of applications.
3. Greater emphasis on modelling.
4. Integration with algebra.
5. Expansion of graphical representations.

Nevertheless, many traditional sequences concerning the sine and cosine rules remained largely unchanged.

As a result, contemporary textbook structures often reflect historical continuity rather than evidence-based pedagogical design.

2.8 Evolution of Triangle-Solving Instruction

2.8.1 Traditional Sequence

Most modern textbooks introduce solving triangles using a progression similar to:

1. Sine Rule (A.S.A./A.A.S.)
2. Sine Rule (S.S.A Ambiguous Case)
3. Cosine Rule (S.A.S.)
4. Cosine Rule (S.S.S.)

This sequence appears logical because it follows increasingly complex geometric configurations.

The original reflection paper observed that this sequence dominates Hong Kong mathematics textbooks and educational materials.

2.8.2 Hidden Pedagogical Assumptions

The traditional sequence rests on several assumptions:

- Mathematical simplicity equals pedagogical simplicity.
- Students can distinguish ambiguity immediately.
- Multiple-solution cases promote conceptual understanding.
- Formula-based organization supports learning.

These assumptions are seldom questioned explicitly.

However, classroom experience frequently reveals that students struggle most with precisely the topics introduced earliest in the sequence, particularly the ambiguous case associated with the sine rule.

2.8.3 Alternative Sequencing Possibilities

Historical development suggests that alternative organizational structures are possible.

Instruction might instead begin with:

- area formulae;
- geometric relationships;
- exact-value reasoning;
- method comparison;
- strategic decision-making.

Interestingly, the original reflection proposes deriving the sine rule through the area formula of a triangle. This approach mirrors the historical reality that geometric reasoning often preceded symbolic formulae.

Consequently, history provides support for considering alternative pedagogical pathways rather than treating conventional textbook sequences as inevitable.

2.9 Trigonometry Education in Hong Kong

Hong Kong mathematics education has inherited influences from British curriculum traditions, examination-oriented assessment structures, and local pedagogical practices.

Trigonometry occupies an important role in the senior secondary curriculum because it supports:

- advanced mathematics;
- physics education;
- engineering preparation;
- problem-solving development.

Textbooks generally emphasize procedural competence and examination performance. As observed in the original paper, substantial instructional time is often devoted to exercises involving the repeated application of formulae.

While this approach can produce strong procedural fluency, questions remain concerning its effectiveness in promoting deeper conceptual understanding.

The examination examples analysed in the original article reveal that even students who possess computational skills may encounter difficulties involving geometric interpretation and ambiguity.

Such findings suggest that curriculum sequencing deserves greater pedagogical attention.

2.10 Summary

The history of trigonometry demonstrates that mathematical ideas evolved gradually across numerous civilizations and practical contexts. From Egyptian surveying and Babylonian astronomy to Greek geometry, Indian sine functions, Islamic triangle-solving methods, and modern educational systems, trigonometry has undergone continual refinement.

Several important themes emerge from this historical review.

First, trigonometry originated from practical problem-solving rather than formula memorization.

Second, geometric reasoning historically preceded symbolic rule application.

Third, mathematical development did not occur in the instructional order commonly used in modern textbooks.



Fourth, curriculum sequences are products of historical and institutional choices rather than unavoidable necessities.

These observations provide an important foundation for the chapters that follow. If instructional sequences are historically contingent rather than mathematically predetermined, then they may legitimately be reconsidered. The central question therefore becomes not whether alternative sequences are possible, but whether alternative sequences may better support student understanding.

The next chapter reviews contemporary research literature concerning trigonometric learning, student misconceptions, procedural fluency, conceptual understanding, and curriculum sequencing in mathematics education.

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Chapter 3

Literature Review

3.1 Introduction

The teaching and learning of trigonometry has attracted considerable attention within mathematics education research over the past four decades. Despite its importance in secondary and post-secondary mathematics, trigonometry remains one of the most challenging topics for students. Researchers have consistently reported difficulties related to conceptual understanding, procedural fluency, selection of appropriate solution methods, interpretation of geometric diagrams, and transfer of knowledge across different problem contexts.

Among the various topics within trigonometry, solving non-right-angled triangles using the sine and cosine formulae occupies a distinctive position. Students must integrate geometric reasoning, algebraic manipulation, symbolic representation, and calculator technology simultaneously. As a result, many learners develop procedural competence without attaining a deep understanding of the relationships among sides, angles, and trigonometric functions.

This chapter reviews the literature relevant to the present study. It examines major strands of research concerning trigonometric learning, conceptual and procedural knowledge, student misconceptions, curriculum sequencing, assessment, calculator usage, and pedagogical approaches. Particular attention is given to findings that can inform the reconsideration of the traditional sequence used for teaching sine and cosine formulae in secondary schools.

3.2 The Nature of Trigonometric Understanding

3.2.1 Trigonometry as a Conceptually Complex Topic

Trigonometry differs from many earlier mathematical topics because it requires students to coordinate multiple mathematical representations simultaneously.

When solving a non-right-angled triangle, students must often:

- interpret a geometric diagram;
- identify known and unknown quantities;
- recognize relationships among sides and angles;
- select an appropriate formula;
- perform algebraic manipulations;
- apply calculator technology correctly;
- evaluate the reasonableness of solutions.

Researchers have argued that trigonometry represents a significant cognitive transition because learners must move beyond concrete arithmetic reasoning toward abstract relational thinking.

Whereas arithmetic emphasizes numbers and operations, trigonometry requires students to understand functions, ratios, geometric relationships, and symbolic representation concurrently.

Consequently, students who appear successful in earlier mathematics topics may encounter difficulties when studying trigonometry for the first time.

3.2.2 Multiple Representations in Trigonometry

A recurring theme in mathematics education research concerns the role of multiple representations.

In trigonometry, understanding may involve:

1. geometric representations;
2. numerical representations;
3. algebraic representations;
4. graphical representations;
5. verbal explanations.

Expert mathematicians move flexibly among these representations. Students, however, frequently become confined to a single representation.

For example, some learners view the sine formula merely as an algebraic equation:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

without recognizing its geometric meaning.

Others may understand the geometric diagram but struggle to translate that understanding into algebraic relationships.

The ability to coordinate these representations is therefore widely regarded as a hallmark of genuine trigonometric understanding.

3.3 Student Difficulties in Learning Trigonometry

3.3.1 Memorization Without Understanding

One of the most frequently reported problems is excessive reliance on memorization.

Many students attempt to succeed in trigonometry by memorizing:

- formulas;
- procedures;
- calculator steps;
- examination routines.

While memorization may support short-term success, it often fails when students encounter unfamiliar problems. Researchers have repeatedly observed that students who rely exclusively on formula recall struggle when required to justify solutions, compare methods, or solve non-routine tasks.

The challenge becomes particularly evident in solving triangles because both the sine formula and cosine formula may be applicable in some situations.

Students must therefore understand not merely how to use a formula but why a formula should be selected.

3.3.2 Difficulty Distinguishing Sides and Angles

Many learners experience confusion when identifying corresponding sides and angles.

For instance, students may incorrectly associate:

- side a with angle B ,
- side b with angle C ,
- side c with angle A .

Such errors indicate a lack of structural understanding of triangles.

Research suggests that students often attend to superficial diagram features rather than underlying mathematical relationships.

Consequently, they may substitute values mechanically into formulas without recognizing whether those values represent corresponding quantities.

3.3.3 Formula Selection Problems

Another commonly reported difficulty concerns method selection.

Students frequently ask:

- Should I use the sine formula?
- Should I use the cosine formula?
- Should I apply both?
- Which method is easier?

From the perspective of experts, these decisions may appear obvious.

For novices, however, selecting an appropriate method represents a substantial cognitive challenge.

This issue is particularly relevant to the present study because sequencing influences method selection.

If students encounter numerous special cases before understanding the broader structure of triangle solving, they may perceive each triangle type as a separate topic rather than as part of an integrated system.

3.4 Conceptual Knowledge and Procedural Knowledge

3.4.1 Defining Conceptual Knowledge

Conceptual knowledge refers to understanding the relationships among mathematical ideas.

A student possessing conceptual understanding of the sine formula recognizes:

- how it is derived;
- why it works;
- when it applies;
- its relationship to area formulas;
- its limitations.

Conceptual understanding enables flexible transfer to new situations.

Students can adapt their knowledge when presented with unfamiliar problems.

3.4.2 Defining Procedural Knowledge

Procedural knowledge refers to the ability to execute algorithms and solution processes.

Examples include:

- substituting values correctly;
- manipulating equations;
- operating calculators;
- obtaining numerical answers efficiently.

Procedural knowledge is important because mathematical competence requires accurate execution.

However, procedural success alone does not guarantee understanding.

Many students can apply formulas correctly while possessing limited insight into their underlying meaning.

3.4.3 The Relationship Between Conceptual and Procedural Knowledge

Educational researchers increasingly argue that conceptual and procedural knowledge should not be viewed as competing goals.

Instead, effective learning emerges when:

- conceptual understanding supports procedures;
- procedural fluency reinforces concepts.

The original reflection paper implicitly adopts this perspective.

For example, deriving the sine formula from the area formula does not merely provide a proof; it establishes conceptual connections that strengthen procedural understanding during problem solving.

Similarly, comparing the sine formula and cosine formula encourages students to understand relationships among methods rather than memorizing isolated procedures.

3.5 Research on the Sine Rule

3.5.1 Advantages of the Sine Rule

The sine rule is generally regarded as one of the first major tools introduced for solving non-right-angled triangles.

Educational researchers note several advantages:

- relatively straightforward algebraic structure;
- direct relationship between sides and opposite angles;
- efficient calculation of unknown quantities;
- strong connections with area formulas.

These characteristics make the sine rule attractive for introductory instruction.

3.5.2 The Ambiguous Case

Despite its simplicity, the sine rule introduces one of the most challenging concepts in school trigonometry: the ambiguous case.

When students solve equations of the form

$$\sin \theta = k$$

two angles may satisfy the condition:

$$\theta$$

and

$$180^\circ - \theta$$

This phenomenon frequently surprises learners.

Research consistently identifies ambiguity as a major source of misconceptions.

Common student errors include:

- accepting only the calculator-generated angle;

- ignoring supplementary angles;
- reporting impossible triangles;
- overlooking multiple solutions.

The original reflection paper documents exactly these classroom difficulties and argues that traditional instructional approaches often overwhelm average students with excessive case analysis.

3.5.3 Alternative Treatments of the Ambiguous Case

Researchers and practitioners have proposed several approaches.

Some advocate:

- extensive classification of cases;
- geometric proofs;
- height-based arguments.

Others recommend:

- visual exploration;
- dynamic geometry software;
- emphasis on geometric reasoning.

The original reflection adopts a simplified pedagogical strategy. Rather than emphasizing numerous geometric cases initially, students are encouraged to focus on interpreting the solutions produced by the sine formula and evaluating their geometric validity.

This approach aligns with contemporary attempts to reduce unnecessary cognitive load.

3.6 Research on the Cosine Rule

3.6.1 Conceptual Importance

The cosine rule occupies a unique position because it generalizes the Pythagorean theorem.

Researchers have shown that students often understand the cosine rule more deeply when its connection to the Pythagorean theorem is explicitly highlighted.

Rather than introducing the formula as an entirely new procedure, effective instruction frames it as an extension of existing knowledge.

3.6.2 Computational Complexity

A recurring theme in the literature concerns computational burden.

The cosine formula:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

often requires more extensive algebraic manipulation than the sine formula.

Consequently, students may perceive it as more difficult.

The original reflection paper notes that repeated applications of the cosine formula may become unnecessarily cumbersome, particularly for students who lack sophisticated calculator support.

This observation raises important questions regarding efficiency and instructional design.

3.6.3 Exact Values and Algebraic Thinking

An advantage of the cosine formula is its ability to produce exact solutions.

When the third side is required in exact surd form, the cosine formula often provides a more elegant approach than numerical approximation.

The original paper argues that this feature is insufficiently emphasized in many classrooms and that students should recognize circumstances in which cosine-rule methods offer clear advantages.

This position reflects broader research emphasizing strategic mathematical thinking.

3.7 Student Misconceptions in Solving Triangles

3.7.1 Misinterpretation of Inverse Trigonometric Functions

Research repeatedly identifies misunderstanding of inverse trigonometric functions as a major obstacle.

Students frequently believe that:

$$\sin^{-1}(k)$$

produces the only possible angle.

This misconception becomes a source of error in S.S.A problems.

The examination example described in the original reflection vividly illustrates this issue, as many students failed to recognize the geometrically appropriate obtuse angle.

3.7.2 Failure to Interpret Geometric Context

Another widely documented problem involves neglecting diagram information.

Students may obtain a numerical answer and immediately stop, without asking:

- Does the answer make sense?
- Is the angle acute or obtuse?
- Is the triangle physically possible?

Researchers increasingly emphasize the importance of mathematical sense-making and geometric interpretation.

3.7.3 Overreliance on Algorithms

Students often develop an algorithmic mindset characterized by:

1. Identify formula.
2. Substitute values.
3. Calculate answer.
4. Move on.

While this approach may work for routine exercises, it often breaks down when non-routine reasoning is required.

The original paper provides an example in which a student solved a comparison problem through geometric reasoning rather than numerical calculation. Such solutions demonstrate deeper mathematical understanding.

3.8 Calculator Technology and Trigonometry Education

3.8.1 Changing Nature of Computation



The widespread availability of scientific calculators has fundamentally transformed trigonometry instruction.

Historically, students devoted substantial effort to numerical computation.

Today, many calculations can be completed instantly.

This shift creates opportunities to focus more heavily on:

- conceptual understanding;
- interpretation;
- reasoning;
- strategy selection.

3.8.2 New Sources of Error

Technology also introduces new challenges.

Students may:

- use incorrect calculator modes;
- misinterpret outputs;
- accept results uncritically;
- become dependent on technology.

The ambiguous case provides a clear example.

Although calculators return a principal angle, students must still determine whether additional solutions exist.

Thus, technology does not eliminate mathematical reasoning; it changes its nature.

3.9 Curriculum Sequencing in Mathematics Education

3.9.1 Why Sequence Matters

Curriculum research increasingly recognizes sequencing as an important component of instructional design.

The order in which concepts are introduced influences:

- cognitive load;
- retention;
- conceptual connections;
- transferability.

A mathematically logical sequence is not necessarily a pedagogically optimal sequence.

3.9.2 Traditional Logic-Based Sequencing

Many mathematics textbooks are organized according to logical structure.

Concepts are presented in an order that reflects formal mathematical relationships.

While this approach offers coherence, it may not always align with how students learn.

Researchers have consequently advocated greater attention to cognitive and developmental considerations.

3.9.3 Cognitive Sequencing

Alternative approaches emphasize:

- reducing complexity;

- building conceptual networks;
- leveraging prior knowledge;
- postponing difficult ideas until foundations are secure.

The original reflection paper exemplifies this perspective by questioning whether the ambiguous case should occupy such an early position in instruction and by proposing stronger links among area formulas, sine formulas, and cosine formulas.

Such proposals align closely with contemporary curriculum research.

3.10 Assessment and Trigonometric Understanding

3.10.1 Limitations of Traditional Assessment

Traditional examinations often emphasize procedural accuracy.

Students receive credit for obtaining correct numerical answers.

However, such assessments may fail to capture:

- conceptual understanding;
- strategic thinking;
- mathematical reasoning.

As a result, students may prioritize memorization over understanding.

3.10.2 Value of Error Analysis

Recent research highlights the importance of analysing student errors.

Errors reveal:

- misconceptions;
- incomplete reasoning;
- hidden assumptions;
- conceptual gaps.

The original reflection paper's examination script analysis provides an example of this approach. Rather than viewing errors merely as failures, the analysis uses them to illuminate student thinking and guide instructional improvement.

3.11 Gaps in Existing Literature

Several gaps emerge from the reviewed literature.

First, much research focuses on student misconceptions while devoting relatively little attention to curriculum sequencing.

Second, studies frequently examine individual concepts rather than the overall structure of solving triangles.

Third, few investigations explore how alternative sequences might reduce ambiguity and cognitive load simultaneously.

Fourth, there remains limited discussion concerning the strategic integration of:

- area formulas;
- sine formulas;



- cosine formulas;
- exact solutions;
- geometric reasoning.

The present manuscript seeks to address these gaps by developing a comprehensive pedagogical framework centred on the sequencing of trigonometric formulae.

3.12 Summary

This chapter has reviewed major research findings concerning trigonometric learning and teaching. The literature reveals that students encounter persistent difficulties involving conceptual understanding, procedural fluency, ambiguity, formula selection, geometric interpretation, and technology use.

Several themes are especially relevant to the present study:

1. Trigonometry requires coordination of multiple representations.
2. Students frequently rely on memorization rather than understanding.
3. The ambiguous case is a major source of misconceptions.
4. Method selection is often difficult for learners.
5. Calculator technology changes but does not eliminate reasoning requirements.
6. Curriculum sequencing plays a significant role in learning outcomes.
7. Greater integration among mathematical ideas may support deeper understanding.

Collectively, these findings provide strong justification for reconsidering how sine and cosine formulae are introduced in secondary education. The next chapter therefore establishes the theoretical framework through which the proposed alternative pedagogical sequence will be analysed. Specifically, constructivism, cognitive load theory, APOS theory, variation theory, and theories of relational understanding will be used to evaluate both conventional and alternative approaches to teaching trigonometric formulae.

Chapter 4

Theoretical Framework

4.1 Introduction

The preceding chapters have established the rationale for re-examining the pedagogical sequence of trigonometric formulae and reviewed relevant literature concerning student difficulties, conceptual understanding, procedural fluency, and curriculum sequencing. The next step is to develop a theoretical framework capable of evaluating both conventional and alternative approaches to teaching the sine and cosine formulae.

Educational practice is often shaped by implicit assumptions about how learning occurs. Some instructional approaches emphasize direct transmission of knowledge, while others prioritize active construction of understanding. Some focus on procedural mastery, whereas others emphasize conceptual development. The selection and sequencing of mathematical content are therefore never purely technical decisions; they are influenced by underlying theories of learning.

The alternative pedagogical approach proposed in this manuscript is informed by several complementary theoretical perspectives. These perspectives provide lenses through which classroom practices can be interpreted and evaluated. Specifically, this chapter examines:

1. Constructivist Learning Theory;
2. Cognitive Load Theory;
3. APOS Theory;
4. Variation Theory;
5. Instrumental and Relational Understanding;
6. Mathematical Connections Theory;
7. Metacognition and Strategic Competence.

Together, these perspectives help explain why certain features of the conventional trigonometry sequence may challenge learners and why alternative sequencing may promote deeper understanding.

4.2 Constructivism and Trigonometric Learning

4.2.1 Foundations of Constructivism

Constructivism is one of the most influential learning theories in contemporary mathematics education. The central premise of constructivism is that learners actively construct knowledge rather than passively receive information.

From a constructivist perspective, understanding develops through interactions between:

- prior knowledge;
- new experiences;
- reflection;
- social discussion;
- problem-solving activities.

Learning therefore involves the continuous reorganization of existing mental structures.

In mathematics classrooms, students do not simply absorb formulas as finished products. Instead, they attempt to make sense of new ideas using previously acquired knowledge.

For trigonometry, prior knowledge may include:

- angle properties;
- similarity of triangles;
- Pythagoras' theorem;
- area of triangles;
- right-angled trigonometric ratios.

These concepts become building blocks for understanding the sine and cosine formulae.

4.2.2 Constructivist Interpretation of Trigonometric Formulae

A constructivist view suggests that students understand new formulas more effectively when they perceive them as natural extensions of existing concepts.

This observation is particularly relevant to the original reflection paper, which proposes introducing the area formula of an S.A.S. triangle before introducing the sine formula. The sine formula is then derived from a geometric relationship already familiar to students rather than presented as an isolated result. This creates a conceptual bridge between previous and new knowledge.

Under this approach:

- known concepts support new concepts;
- relationships become visible;
- formulas appear meaningful;
- memorization demands are reduced.

Constructivist theory therefore supports instructional sequences that emphasize derivation, connection, and sense-making.

4.2.3 Implications for Curriculum Sequencing

If knowledge is actively constructed, instructional order becomes critically important.

Students are more likely to understand a concept when:

- prerequisite ideas have been established;
- cognitive connections are explicit;
- complexity develops gradually.

Consequently, curriculum sequencing should be evaluated not solely according to mathematical logic but also according to how learners build meaning.

The alternative pedagogical framework proposed later in this manuscript is consistent with this constructivist principle.

4.3 Cognitive Load Theory

4.3.1 Overview of Cognitive Load Theory

Cognitive Load Theory (CLT) provides a powerful explanation for why students sometimes struggle with mathematically sophisticated procedures.

According to CLT, working memory possesses limited capacity. When instructional materials require learners to process too much information simultaneously, learning becomes less effective.

Three types of cognitive load are commonly distinguished:

Intrinsic Load

Intrinsic load refers to the inherent complexity of a task.

In solving triangles, intrinsic load may arise from:

- multiple variables;
- geometric relationships;
- formula selection;
- angle-side correspondences.

Some of this complexity cannot be eliminated because it is fundamental to the topic itself.

Extraneous Load

Extraneous load refers to unnecessary cognitive demands created by instructional design.

Examples include:

- complicated notational systems;
- excessive case distinctions;
- poorly structured explanations;
- unnecessary calculations.

Extraneous load consumes cognitive resources without contributing meaningfully to learning.

Germane Load

Germane load refers to mental effort devoted to constructing useful knowledge structures.

Examples include:

- recognizing patterns;
- comparing methods;
- understanding derivations;
- explaining reasoning.

This form of cognitive effort supports meaningful learning.

4.3.2 Cognitive Load and the Ambiguous Case

The S.S.A ambiguous case provides an excellent example of cognitive load considerations.

Traditional textbook treatments often require students to consider:

- acute angles;
- obtuse angles;
- right angles;
- different side-length comparisons;

- multiple diagrams;
- several possible outcomes.

Although mathematically rigorous, this approach can produce significant cognitive burden.

The original reflection paper argues that many average students experience these cases as confusing and that the resulting complexity creates unnecessary "noise."

From a Cognitive Load Theory perspective, this observation can be interpreted as excessive extraneous load.

Students may become preoccupied with remembering categories rather than understanding fundamental relationships.

4.3.3 Reducing Extraneous Cognitive Load

An alternative instructional sequence should aim to:

- reduce unnecessary complexity;
- postpone advanced distinctions until appropriate;
- emphasize core concepts first;
- simplify decision-making processes.

The strategy suggested in the original paper focuses students initially on interpreting solutions produced by the sine formula rather than classifying multiple geometric cases.

This approach may free cognitive resources for conceptual understanding.

Consequently, Cognitive Load Theory provides strong support for re-evaluating conventional textbook sequencing.

4.4 APOS Theory

4.4.1 Overview of APOS Theory

APOS Theory was developed to explain how learners construct mathematical concepts.

The acronym APOS represents:

- Action
- Process
- Object
- Schema

Learning is viewed as a progression through increasingly sophisticated levels of understanding.

4.4.2 Action Stage

At the Action stage, learners perform procedures mechanically.

For example, a student using the sine formula may simply substitute values according to a memorized pattern:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

without understanding why the relationship holds.

Many students initially operate at this level.

4.4.3 Process Stage

At the Process stage, students begin to internalize procedures.

They can:

- anticipate results;
- recognize patterns;
- explain steps;
- reverse procedures.

For example, students may understand that equal sine ratios arise from common area relationships within the triangle.

4.4.4 Object Stage

At the Object stage, mathematical ideas become conceptual entities.

Students can treat the sine formula as an integrated mathematical object rather than a sequence of procedural steps.

They begin comparing methods strategically.

Questions such as:

- "Should I use the sine formula?"
- "Would the cosine formula be easier?"

become meaningful.

4.4.5 Schema Stage

At the Schema stage, multiple concepts become interconnected.

Students recognize relationships among:

- area formulas;
- sine formulas;
- cosine formulas;
- congruence conditions;
- exact values;
- ambiguity.

Interestingly, the original reflection paper repeatedly emphasizes such connections rather than treating formulas independently.

This suggests movement toward Schema-level understanding.

4.4.6 APOS Interpretation of Alternative Sequencing

From an APOS perspective, effective teaching should help learners progress from actions toward schemas.

This progression is facilitated when:

- formulas are connected;
- derivations are explained;
- method comparisons are discussed;

- conceptual relationships are emphasized.

The proposed pedagogical sequence is therefore highly compatible with APOS Theory.

4.5 Variation Theory

4.5.1 Core Principles of Variation Theory

Variation Theory argues that learning occurs when critical features of a concept become visible through systematic variation.

To understand a concept fully, learners must experience meaningful differences and contrasts.

Variation helps students identify:

- what changes;
- what remains invariant;
- what features are essential.

4.5.2 Application to Trigonometry

In solving triangles, critical features include:

- given information;
- unknown quantities;
- number of solutions;
- applicable formulas.

Students often fail to understand distinctions because examples appear too similar.

Variation Theory suggests presenting carefully selected examples that highlight:

Example Variation 1

Comparing:

- A.A.S. triangle
- S.S.A triangle

reveals why ambiguity occurs in one case but not the other.

Example Variation 2

Comparing:

- S.S.S. triangle
- S.A.S. triangle

reveals different roles of the cosine formula.

Example Variation 3

Comparing:

- exact-value solutions;
- calculator approximations;

highlights the strengths and limitations of different methods.

4.5.3 Pedagogical Significance

The original reflection frequently compares alternative solution methods and encourages students to discuss their relative efficiency.

Such comparison naturally creates variation.

Instead of presenting one method as universally correct, instruction highlights situations in which different methods are advantageous.

Variation Theory therefore supports deliberate comparison as a pedagogical strategy.

4.6 Instrumental and Relational Understanding

4.6.1 Instrumental Understanding

Instrumental understanding refers to knowing how to perform procedures.

Students with instrumental understanding can often:

- substitute numbers;
- solve equations;
- obtain correct answers.

However, they may be unable to explain why procedures work.

For example, a student may remember:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

without understanding its derivation or interpretation.

4.6.2 Relational Understanding

Relational understanding refers to knowing both:

- what to do;
- why it works.

Students possessing relational understanding recognize connections among ideas.

For trigonometry, this includes understanding relationships among:

- Pythagoras' theorem;
- area formulas;
- sine formulas;
- cosine formulas;
- geometric properties.

4.6.3 Relevance to the Present Study

A major concern expressed in the original reflection is that students often apply formulas mechanically and fail to interpret results geometrically.

The examination examples illustrate this issue clearly.

Students obtained numerical solutions but neglected contextual reasoning.

One notable exception involved a student who solved a comparison problem using the geometric principle:

larger angle $\leftrightarrow$ longer side.

Rather than applying the sine formula unnecessarily, the student demonstrated relational understanding. This example illustrates the educational value of relational thinking.

4.7 Mathematical Connections Theory

4.7.1 Importance of Connected Knowledge

Many mathematics educators argue that learning becomes more durable when concepts are interconnected.

Connections may occur among:

- mathematical topics;
- representations;
- procedures;
- applications.

Knowledge organized as a network is generally more flexible than isolated facts.

4.7.2 Connections Within Trigonometry

The original reflection repeatedly encourages students to connect:

- area formulas and sine formulas;
- cosine formulas and quadratic equations;
- angle properties and triangle solving;
- exact values and algebraic methods.

These connections transform solving triangles from a collection of procedures into a coherent mathematical system.

Such coherence is likely to support both retention and transfer.

4.8 Metacognition and Strategic Competence

4.8.1 What Is Metacognition?

Metacognition refers to awareness and regulation of one's own thinking.

Effective learners ask questions such as:

- Does my answer make sense?
- Is there another method?
- Have I considered all possible solutions?
- Which approach is most efficient?

4.8.2 Strategic Competence in Mathematics

Strategic competence involves:

- selecting methods;
- adapting procedures;
- evaluating results;
- comparing alternatives.

Research indicates that expert mathematicians possess highly developed strategic competence.

4.8.3 Strategic Thinking and Triangle Solving

The original paper frequently invites students to compare:

- sine-formula methods;
- cosine-formula methods;
- exact versus approximate solutions;
- different routes to the same answer.

This emphasis encourages metacognitive reflection.

Students move beyond simply solving problems toward understanding why particular methods may be preferable.

4.9 An Integrated Theoretical Model for This Study

The discussions above suggest that no single theory fully explains trigonometric learning.

Instead, the present study adopts an integrated framework.

Constructivism

Explains how students build new knowledge from prior knowledge.

Cognitive Load Theory

Explains why certain instructional sequences create unnecessary difficulty.

APOS Theory

Explains progression from procedural execution to conceptual integration.

Variation Theory

Explains how carefully structured comparisons promote understanding.

Relational Understanding

Explains the importance of meaningful mathematical connections.

Metacognitive Theory

Explains strategic method selection and self-monitoring.

Together, these theories provide complementary perspectives on trigonometry teaching.

4.10 Summary

This chapter has established the theoretical foundation for evaluating the pedagogical sequence of trigonometric formulae.

Several important conclusions emerge.

First, constructivist theory suggests that students learn more effectively when new formulas are derived from familiar ideas and embedded within meaningful conceptual networks.

Second, Cognitive Load Theory indicates that conventional treatments of the ambiguous S.S.A case may impose unnecessary cognitive burdens on learners.

Third, APOS Theory highlights the importance of progressing from procedural actions toward integrated conceptual schemas.

Fourth, Variation Theory supports systematic comparison of formulas, methods, and problem types.



Fifth, theories of relational understanding emphasize that mathematical proficiency requires more than computational accuracy; it requires recognition of underlying relationships.

Finally, metacognitive and strategic perspectives demonstrate the value of helping students evaluate, compare, and select among alternative solution methods.

Collectively, these theories provide a strong foundation for re-examining the traditional textbook sequence. In the next chapter, these theoretical perspectives will be applied directly to a critical analysis of the conventional instructional sequence used for introducing the sine rule and cosine rule in Hong Kong secondary mathematics education.

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Chapter 5

A Critical Analysis of the Conventional Pedagogical Sequence for Teaching Trigonometric Formulae

5.1 Introduction

The previous chapters have established the historical, theoretical, and research foundations for examining the sequencing of trigonometric instruction. This chapter focuses specifically on the conventional pedagogical sequence commonly adopted in Hong Kong secondary mathematics textbooks and classrooms.

As noted in the original reflection paper, most textbooks introduce the topic of solving non-right-angled triangles in the following order:

1. Sine Formula for A.S.A. and A.A.S. triangles;
2. Sine Formula for S.S.A. triangles (the ambiguous case);
3. Cosine Formula for S.A.S. triangles;
4. Cosine Formula for S.S.S. triangles.

This sequence has become so widely accepted that many teachers regard it as the natural progression for teaching trigonometry. However, educational traditions are not necessarily educationally optimal. The purpose of this chapter is to critically examine the strengths, limitations, assumptions, and unintended consequences of the conventional sequence from the perspectives of mathematics education, cognitive psychology, and classroom practice.

The analysis will demonstrate that although the traditional sequence possesses strong mathematical logic, it may also create avoidable learning difficulties, particularly for average and lower-attaining students.

5.2 The Structure of the Conventional Sequence

5.2.1 Typical Textbook Organization

Most secondary mathematics textbooks organize the topic according to the pattern shown in Figure 5.1.

Stage 1

Introduction of the Sine Formula

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

with straightforward A.S.A. and A.A.S. examples.

Stage 2

Extension to S.S.A. triangles involving:

- no solution;
- one solution;
- two solutions.

Stage 3

Introduction of the Cosine Formula

$$a^2 = b^2 + c^2 - 2bc \cos A$$

for S.A.S. situations.

Stage 4

Application of the Cosine Formula to S.S.S. triangles.

Stage 5

Mixed exercises involving triangle-solving and word problems.

At first glance, this progression appears systematic and logical.

The sequence progresses:

- from simpler to more complex;
- from one formula to another;
- from direct application to extended application.

However, closer examination reveals several hidden pedagogical assumptions.

5.2.2 Underlying Assumptions

The conventional sequence appears to assume that:

Assumption 1

Mathematical Simplicity Equals Learning Simplicity

Because the sine formula is generally easier to manipulate algebraically than the cosine formula, textbooks often place it first.

Assumption 2

Students Can Handle Ambiguity Early

The ambiguous case is typically introduced immediately after basic sine-rule applications.

Assumption 3

Formula-Based Organization Supports Learning

The topic is organized around formulas rather than concepts.

Assumption 4

Computational Development Should Drive Curriculum Structure

Instruction frequently progresses according to increasing computational sophistication.

These assumptions are rarely stated explicitly but exert considerable influence on classroom practice.

5.3 Strengths of the Conventional Sequence

Before examining limitations, it is important to acknowledge the advantages of the traditional approach.

5.3.1 Strong Mathematical Logic

The traditional sequence reflects the internal structure of trigonometry.

Students first learn:

- one formula;
- applications of that formula;
- a second formula;
- applications of the second formula.

This organization may appear straightforward to both teachers and curriculum developers.

Furthermore, the sequence allows students to focus on one major formula at a time.

This can simplify lesson planning and assessment design.

5.3.2 Alignment with Examination Questions

Many public examinations have historically used a progression similar to textbook presentations.

Consequently, the traditional sequence aligns well with examination preparation.

Students become familiar with:

- common question formats;
- standard procedures;
- typical solution pathways.

Teachers often appreciate the predictability this structure provides.

5.3.3 Efficiency for High-Attaining Students

Highly capable students frequently succeed regardless of instructional sequence.

Such learners are often able to:

- identify patterns independently;
- integrate concepts spontaneously;
- manage procedural complexity.

For these students, the conventional sequence may work reasonably well.

Indeed, many mathematically successful individuals were taught using precisely this approach.

However, educational effectiveness should be evaluated not only by its impact on the strongest students but also by its accessibility to the broader student population.

5.4 The Early Introduction of the Ambiguous Case

5.4.1 A Central Pedagogical Concern

One of the most significant concerns with the conventional sequence is the early introduction of the S.S.A. ambiguous case.

From a purely mathematical perspective, introducing the ambiguous case immediately after the sine formula appears logical because it represents another application of the same formula.

From a pedagogical perspective, however, the situation is less clear.

Students who have only recently learned:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

are suddenly confronted with the possibility that:

- one answer may be correct;
- two answers may be correct;
- no answer may exist.

This represents a substantial increase in conceptual complexity.

5.4.2 Cognitive Load Implications

As discussed in Chapter 4, Cognitive Load Theory distinguishes between intrinsic and extraneous cognitive load.

When students encounter the ambiguous case, they must simultaneously consider:

- side-angle correspondence;
- inverse trigonometric functions;
- supplementary angles;
- angle-sum properties;
- triangle validity conditions;
- geometric interpretation.

The number of interacting concepts increases dramatically.

Many students consequently become overwhelmed.

The original reflection paper explicitly notes that textbook treatments often generate excessive "noise" for average students because of the numerous cases that must be considered.

From a cognitive-load perspective, this criticism is highly significant.

The issue is not that the mathematics is incorrect.

Rather, the issue is whether the timing of the topic is pedagogically appropriate.

5.4.3 Procedural Learning Without Understanding

A common classroom outcome is that students memorize a series of case distinctions.

For example:

- if side is longer than height ...
- if side equals height ...
- if side shorter than height ...

Students often memorize these procedures without understanding the underlying geometry.

Such learning tends to be fragile.

When faced with unfamiliar problems, students frequently revert to guessing.

5.5 Formula-Centred Rather Than Concept-Centred Instruction

5.5.1 Organization Around Procedures

A notable feature of many textbooks is that instruction is organized around formulas.

Students encounter chapters entitled:

- Sine Rule;
- Cosine Rule;
- Applications of Sine Rule;
- Applications of Cosine Rule.

Although convenient, this structure may unintentionally encourage students to view formulas as independent topics.

As a result, students often fail to perceive relationships among concepts.

5.5.2 Fragmented Knowledge Structures

When formulas are introduced separately, learners may construct fragmented knowledge.

For example:

**Mental Compartment A**

Sine Formula

Mental Compartment B

Cosine Formula

Mental Compartment C

Area Formula

Students may never fully appreciate how these ideas are interconnected.

Yet mathematically they are closely related.

One of the most interesting observations in the original paper is the proposal to derive the sine formula from the area formula of a triangle.

Such derivations help reveal conceptual links that are often obscured in traditional instruction.

5.5.3 Consequences for Transfer

Research in mathematics education consistently demonstrates that transfer becomes more difficult when knowledge is compartmentalized.

Students may know:

- when to use a formula in a textbook exercise;

but may struggle with:

- selecting methods in unfamiliar situations;
- comparing alternative approaches;
- evaluating efficiency.

Consequently, formula-centred instruction may inadvertently limit strategic competence.

5.6 The Treatment of the Cosine Formula**5.6.1 Cosine Formula as a Secondary Tool**

In most textbooks, the cosine formula appears only after extensive discussion of the sine formula.

This sequencing can create an unintended impression:

The sine formula is the primary method.

The cosine formula is a secondary method.

However, many classroom situations do not support this distinction.

The cosine formula often offers advantages that students may fail to recognize.

5.6.2 Underutilization of Exact Solutions

The original reflection paper points out that the cosine formula can be particularly useful when exact values of lengths are required.

Traditional instruction tends to emphasize numerical answers and calculator outputs.

As a result, students may overlook opportunities to obtain elegant exact solutions.

This is especially problematic because:

- exact values reinforce algebraic thinking;

- surd expressions strengthen symbolic manipulation;
- exact methods support deeper mathematical understanding.

5.6.3 Repeated Applications of the Cosine Formula

Another feature of conventional instruction involves applying the cosine formula repeatedly.

For example, in S.S.S. triangles:

1. find one angle using cosine formula;
2. find another angle using cosine formula;
3. determine remaining angle.

Although mathematically correct, this process is often computationally demanding.

The original paper argues that repeatedly applying the cosine formula may be unnecessarily complicated for many students.

Alternative approaches involving a strategic combination of cosine and sine formulas may sometimes be more accessible.

5.7 The Overemphasis on Calculation

5.7.1 Calculation as the Dominant Goal

Traditional exercises frequently emphasize obtaining numerical answers.

Students are rewarded for:

- substitution;
- simplification;
- calculator accuracy.

Less attention is often given to:

- reasoning;
- interpretation;
- comparison;
- justification.

5.7.2 Loss of Geometric Sense

One consequence of calculation-dominated instruction is reduced geometric awareness.

Students may successfully compute:

$$\sin^{-1}(0.75) = 48.59037789^\circ$$

without considering whether the required angle should actually be obtuse.

This issue was observed repeatedly in the examination example discussed in the original reflection paper.

The error was not computational.

The error was conceptual.

Students failed to interpret the geometry of the situation.

5.7.3 Missed Opportunities for Mathematical Reasoning



The second examination example in the original paper offers an illuminating contrast.

Most students calculated lengths using the sine formula.

One student instead observed that a larger angle is opposite a longer side and reached the conclusion directly.

This solution demonstrates:

- geometric reasoning;
- efficiency;
- conceptual understanding.

Traditional instructional sequences may not provide enough opportunities for developing such habits of mind.

5.8 Student Errors as Evidence of Pedagogical Weaknesses

5.8.1 Errors Are Diagnostic

Student errors provide valuable information about learning.

Persistent patterns of error often reveal weaknesses in instruction rather than weaknesses in students alone.

5.8.2 Common Errors in the Traditional Sequence

The original reflection paper identifies several recurring difficulties:

Error 1

Ignoring the Second Quadrant Solution

Students accept calculator outputs uncritically.

Error 2

Misinterpreting Diagram Information

Students fail to use geometric context.

Error 3

Overusing Computational Methods

Students calculate unnecessarily when logical reasoning would suffice.

Error 4

Failure to Evaluate Reasonableness

Students rarely ask whether answers make geometric sense.

Taken collectively, these patterns suggest that conventional instruction may prioritize procedures at the expense of understanding.

5.9 Evaluation Through the Theoretical Framework

5.9.1 Constructivist Perspective

From a constructivist viewpoint, traditional sequencing sometimes introduces complexity before conceptual foundations are secure.

Students are expected to manage ambiguity before developing strong conceptual networks.

5.9.2 Cognitive Load Perspective



From a cognitive-load perspective, the ambiguous case introduces substantial cognitive demands early in the learning process.

This may overwhelm working memory and impede understanding.

5.9.3 APOS Perspective

From an APOS perspective, traditional instruction often leaves students at the Action stage.

Students learn procedures but may not construct integrated schemas connecting different formulas.

5.9.4 Variation Theory Perspective

Variation Theory suggests that students benefit from meaningful comparisons.

Many textbook treatments provide examples of the same method repeatedly rather than contrasting multiple approaches.

As a result, critical features may remain invisible.

5.9.5 Relational Understanding Perspective

Traditional instruction often encourages instrumental understanding.

Students learn how to solve triangles but not necessarily why particular methods work or when alternative methods may be preferable.

5.10 Summary

This chapter has critically examined the conventional pedagogical sequence used for teaching the sine and cosine formulae.

Several strengths were identified, including mathematical coherence, examination alignment, and familiarity within educational practice.

However, significant limitations also emerged.

The analysis suggests that:

1. The ambiguous case may be introduced too early.
2. Excessive cognitive load may hinder understanding.
3. Formula-centred organization may fragment knowledge.
4. The cosine formula may be underutilized conceptually.
5. Calculation is often emphasized at the expense of reasoning.
6. Persistent student errors indicate deeper conceptual difficulties.
7. Opportunities for strategic thinking are frequently limited.

These findings do not imply that the conventional sequence should be abandoned entirely. Rather, they suggest that alternative instructional frameworks deserve serious consideration.

The next chapter therefore presents a comprehensive alternative pedagogical sequence inspired by classroom experience, informed by educational theory, and grounded in the principles articulated in the original reflection paper. There, a new model for introducing and integrating the sine formula, cosine formula, area formula, exact values, ambiguity, and geometric reasoning will be proposed and justified in detail.

Chapter 6

An Alternative Pedagogical Framework for Teaching Trigonometric Formulae

6.1 Introduction

The previous chapter critically examined the conventional pedagogical sequence commonly used for introducing the sine and cosine formulae in secondary mathematics education. While the traditional approach possesses mathematical coherence and curricular familiarity, it was argued that several features may create unnecessary challenges for learners, particularly during the early introduction of the ambiguous case and the tendency to treat formulas as isolated procedures rather than interconnected concepts. The original reflection paper likewise questioned whether existing textbook approaches represent the most effective pathway for student learning and proposed several alternative strategies based upon classroom experience.

This chapter presents a comprehensive alternative pedagogical framework for teaching the sine formula and cosine formula. The framework does not reject existing mathematical content, nor does it seek to eliminate important concepts such as the ambiguous case. Rather, it proposes a different instructional sequence that aims to:

- strengthen conceptual understanding;
- reduce unnecessary cognitive burden;
- emphasize mathematical connections;
- promote strategic competence;
- encourage geometric reasoning;
- enhance students' confidence in solving triangles.

The framework is grounded in the principles discussed in Chapters 3 and 4 and in the classroom observations documented in the original reflection paper.

6.2 Guiding Principles of the Alternative Framework

Before describing the sequence itself, it is useful to identify the principles that underpin the proposed approach.

6.2.1 Connections Before Procedures

Students frequently experience mathematics as a collection of disconnected rules.

One of the central aims of the proposed framework is therefore to help students recognize the relationships among:

- area formulae;
- sine formulae;
- cosine formulae;
- congruence conditions;
- triangle properties;
- geometric reasoning.

The learning objective is not merely procedural proficiency but the construction of an integrated conceptual network.

6.2.2 Geometric Meaning Before Algebraic Manipulation

Many students learn formulas without understanding the geometry underlying them.

The proposed framework therefore emphasizes:

- visual reasoning;
- diagram interpretation;
- geometric relationships;

before extensive symbolic manipulation.

Students should understand why a formula exists before learning how to use it.

6.2.3 Reduction of Unnecessary Complexity

The framework seeks to reduce extraneous cognitive load.

This does not mean simplifying mathematics.

Rather, it means avoiding unnecessary complexity during the early stages of learning.

Particularly challenging concepts, such as the ambiguous case, are introduced only after students have developed a strong understanding of the fundamental relationships involved.

6.2.4 Strategic Rather Than Mechanical Thinking

Students are encouraged to ask:

- Why should this method be used?
- Is there a shorter approach?
- Could another formula work?
- Does the answer make sense?

The ultimate goal is to develop mathematically strategic thinkers rather than formula-followers.

6.3 Overview of the Proposed Sequence

The proposed instructional progression differs substantially from the traditional textbook sequence.

Stage 1

Review of Triangle Geometry and Area Relationships

Stage 2

Development of the Area Formula

$$\text{Area} = \frac{1}{2} ab \sin C$$

Stage 3

Derivation and Introduction of the Sine Formula

Stage 4

Applications of the Sine Formula to A.A.S. and A.S.A. Problems

Stage 5

Introduction of the Cosine Formula Through the Generalization of Pythagoras' Theorem

Stage 6

Applications of the Cosine Formula to S.A.S. and S.S.S. Problems

Stage 7

Comparison Between Sine-Formula and Cosine-Formula Methods

Stage 8

Introduction of the Ambiguous S.S.A. Case

Stage 9

Advanced Problem Solving and Strategic Method Selection

The most significant difference is that the ambiguous case is postponed until students possess greater conceptual maturity and a more complete toolkit.

6.4 Stage 1: Revisiting Geometric Foundations**6.4.1 Why Revisit Geometry?**

Many students enter the topic of solving triangles with fragile geometric understanding.

They may remember formulas but forget key principles such as:

- angle sum of a triangle;
- larger angle opposite longer side;
- congruence conditions;
- properties of isosceles triangles.

These ideas are critically important for later reasoning.

Consequently, the proposed sequence begins with a review of essential geometric principles.

6.4.2 Emphasis on Qualitative Reasoning

Rather than immediately calculating answers, students are encouraged to answer questions such as:

- Which side is longest?
- Which angle is largest?
- Can this triangle exist?
- Which lengths appear reasonable?

Such discussions strengthen geometric intuition.

The importance of this intuition is illustrated in the second examination example described in the original reflection paper, where a student successfully solved a problem using the principle that larger angles are opposite longer sides without performing detailed calculations.

6.5 Stage 2: Introducing the Area Formula**6.5.1 The Area Formula as a Bridge Concept**

In the traditional sequence, the sine formula is often introduced directly.

The proposed sequence instead begins with:

$$\text{Area} = \frac{1}{2}ab \sin C$$

This formula possesses several advantages.

First, it connects directly with earlier knowledge concerning:



- area of triangles;
- heights;
- right-angled trigonometry.

Second, it provides a meaningful geometric context.

Third, it serves as a bridge to the sine formula.

6.5.2 Conceptual Derivation

Students derive the formula using existing knowledge.

Consider a triangle with sides a , b , and included angle C .

The height can be expressed as:

$$h = b \sin C$$

Therefore:

$$\text{Area} = \frac{1}{2} a(b \sin C)$$

which gives:

$$\text{Area} = \frac{1}{2} ab \sin C$$

This derivation reinforces previously learned concepts and supports constructivist learning.

6.5.3 Educational Benefits

The area formula helps students appreciate that:

- trigonometry is not simply about ratios;
- sine functions arise naturally in geometry;
- formulas are connected rather than isolated.

These insights prepare students for the next stage.

6.6 Stage 3: Deriving the Sine Formula

6.6.1 Derivation Rather Than Presentation

The original reflection paper proposes introducing the sine formula through the area formula.

This idea forms a cornerstone of the alternative framework.

Instead of presenting the sine formula as a result to memorize, students develop it themselves.

6.6.2 Derivation

For a triangle:

$$\text{Area} = \frac{1}{2} bc \sin A$$

$$\text{Area} = \frac{1}{2} ca \sin B$$



$$\text{Area} = \frac{1}{2} ab \sin C$$

Equating these expressions yields:

$$bc \sin A = ca \sin B$$

and

$$ca \sin B = ab \sin C$$

leading to:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Students therefore understand not only the formula but also its origin.

6.6.3 Pedagogical Advantages

Such derivations:

- reduce memorization demands;
- strengthen conceptual understanding;
- reveal mathematical structure;
- build confidence.

The formula becomes meaningful rather than arbitrary.

6.7 Stage 4: Initial Applications of the Sine Formula

6.7.1 Restricting Early Applications

At this stage, the sine formula is applied only to:

- A.A.S. triangles;
- A.S.A. triangles.

The ambiguous S.S.A. case is deliberately excluded.

This restriction reduces cognitive complexity and allows students to master fundamental ideas first.

6.7.2 Focus on Length Determination

As suggested in the original reflection paper, initial emphasis should be placed on using the sine formula to find unknown lengths because these situations produce unique solutions and involve relatively little ambiguity.

Students can therefore concentrate on understanding the meaning of the formula.

6.8 Stage 5: Introducing the Cosine Formula

6.8.1 Cosine Formula as an Extension of Pythagoras

The cosine formula should not be introduced as an entirely new topic.

Instead, it should be presented as a generalization of:

$$a^2 + b^2 = c^2$$

This approach highlights continuity between old and new knowledge.

6.8.2 Geometric Derivation

The derivation illustrates that the cosine formula extends familiar geometric relationships beyond right triangles.

Students see that:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

is fundamentally related to concepts they already understand.

This strengthens relational understanding.

6.8.3 Conceptual Positioning

The cosine formula becomes:

- a tool for lengths;
- a tool for exact values;
- a tool for removing ambiguity.

Students begin to appreciate its strategic role.

6.9 Stage 6: Solving S.A.S. and S.S.S. Triangles

6.9.1 S.A.S. Triangles

Students first use the cosine formula to determine the third side.

Subsequently, they may:

- use the sine formula; or
- use the cosine formula again.

A discussion of comparative efficiency is encouraged.

This reflects the recommendation made in the original reflection paper.

6.9.2 S.S.S. Triangles

Instead of applying the cosine formula repeatedly without reflection, students examine multiple methods.

Questions include:

- Which angle should be found first?
- Which method is simplest?
- Which method minimizes ambiguity?

These discussions cultivate strategic thinking.

6.9.3 Emphasis on Method Comparison

Students compare:

Method A

Repeated cosine-formula applications.

Method B

One cosine-formula application followed by sine-rule reasoning.

Method C

Geometric argument where appropriate.

Such comparisons embody Variation Theory and metacognitive principles.

6.10 Stage 7: Explicit Comparison of Formulae

6.10.1 Why Compare Methods?

Textbooks often teach formulas separately.

The proposed framework instead encourages direct comparison.

Students investigate:

Feature	Sine Formula	Cosine Formula
Unknown side	Strong	Strong
Unknown angle	Strong	Strong
Exact values	Limited	Strong
Ambiguity	Possible	Rare
Algebraic complexity	Lower	Higher

The objective is not memorization but strategic awareness.

6.10.2 Developing Mathematical Judgment

Students begin to recognize that good mathematics involves choosing methods rather than simply following instructions.

This is a key characteristic of expert problem solvers.

6.11 Stage 8: Introducing the Ambiguous Case

6.11.1 Delayed Introduction

Only after students understand:

- sine formula;
- cosine formula;
- geometric reasoning;
- triangle structure;

is the ambiguous case introduced.

This timing differs fundamentally from the conventional sequence.

6.11.2 Reframing the Problem

Instead of classifying numerous special cases immediately, students first investigate:

$$\sin \theta = k$$

and observe that:

$$\theta$$

and

$$180^\circ - \theta$$

have equal sine values.



The ambiguity therefore emerges naturally.

6.11.3 Use of Cosine Formula as a Comparison Tool

The original reflection paper notes that the cosine formula can provide an alternative method for understanding S.S.A. situations.

Students can investigate:

- one solution;
- two solutions;
- no solution;

from both geometric and algebraic perspectives.

This deeper exploration promotes understanding rather than memorization.

6.12 Stage 9: Strategic Problem Solving

6.12.1 Beyond Formula Application

The final phase focuses on genuine problem solving.

Students are encouraged to ask:

- Which method is best?
- Which method is shortest?
- Which method provides exact values?
- Which method explains the situation most clearly?

6.12.2 Encouraging Multiple Solutions

Problems should allow multiple approaches.

For example:

- sine formula;
- cosine formula;
- geometric arguments;
- symmetry;
- triangle properties.

This flexibility develops mathematical maturity.

6.12.3 Error Analysis Activities

Students examine common mistakes such as:

- neglecting supplementary angles;
- choosing incorrect formulas;
- ignoring geometric information.

The examination examples discussed in the original reflection paper provide valuable teaching resources for such activities.

6.13 Expected Benefits of the Alternative Framework

The proposed sequence is expected to provide several advantages.

Conceptual Benefits

- Stronger understanding of derivations.
- Better appreciation of connections.
- Improved geometric reasoning.

Cognitive Benefits

- Reduced extraneous cognitive load.
- More gradual development of complexity.
- Improved retention.

Strategic Benefits

- Better method selection.
- More flexible problem solving.
- Increased metacognitive awareness.

Affective Benefits

- Greater confidence.
- Reduced anxiety.
- Higher engagement.

6.14 Potential Limitations

The proposed framework is not without challenges.

Potential limitations include:

- increased instructional time;
- curriculum coverage pressures;
- examination demands;
- teacher readiness.

Teachers accustomed to traditional approaches may initially find the alternative sequence unfamiliar.

Nevertheless, such challenges are manageable through professional development and thoughtful curriculum planning.

6.15 Summary

This chapter has presented an alternative pedagogical framework for introducing the sine formula and cosine formula in secondary mathematics education. Building upon the classroom insights of the original reflection paper and informed by contemporary educational theory, the framework proposes a sequence that prioritizes conceptual understanding, mathematical connections, strategic thinking, and reduced cognitive burden.

The central innovations include:

1. Introducing the area formula before the sine formula.
2. Deriving formulas rather than merely presenting them.
3. Treating the cosine formula as a conceptual partner rather than a secondary tool.



4. Delaying the ambiguous case until conceptual foundations are secure.
5. Encouraging systematic comparison of alternative methods.
6. Promoting strategic problem solving and geometric reasoning.

Having established the structure of the proposed framework, the next chapter examines how such an approach might be implemented in practice through a preliminary classroom assessment based on reflective teaching experiences in a Hong Kong secondary school.

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Chapter 7

Preliminary Classroom Assessment of the Alternative Pedagogical Sequence in a Hong Kong Secondary School

7.1 Introduction

The previous chapter proposed an alternative pedagogical framework for teaching the sine formula and cosine formula in secondary mathematics. While the framework was informed by educational theory and reflective practice, an important question remains:

How might such an approach operate in actual classrooms?

The present study is conceptual rather than empirical. Consequently, this chapter does not claim to present a large-scale experimental investigation. Instead, it draws upon the classroom observations, examination experiences, instructional practices, and reflective teaching insights documented in the original paper to conduct a preliminary assessment of the proposed approach.

The purpose of this chapter is threefold:

1. To examine how the alternative sequence may influence student learning.
2. To analyze observed student responses and common misconceptions.
3. To evaluate the pedagogical strengths and limitations of the proposed framework within the context of a Hong Kong secondary school.

Rather than presenting statistical evidence, the chapter provides a practitioner-oriented assessment grounded in classroom experience.

7.2 Context of the Classroom Assessment

7.2.1 School Context

The observations discussed in this chapter emerged from the teaching of senior secondary mathematics in a Hong Kong secondary school. The topic under consideration formed part of the curriculum section commonly known as "More About Trigonometry," which includes:

- sine formula;
- cosine formula;
- solving triangles;
- applications of trigonometry;
- geometrical problem solving.

Students entering this topic had previously studied:

- angle properties;
- congruent triangles;
- similar triangles;
- Pythagoras' theorem;
- right-angled trigonometry.

Consequently, they possessed many of the prerequisite concepts necessary for learning non-right-angled trigonometry.

7.2.2 Characteristics of Learners

Like many Hong Kong mathematics classrooms, student ability levels varied considerably.

Broadly speaking, three categories of learners could be identified:

High-Achieving Students

These students typically:

- learned formulas quickly;
- manipulated algebra confidently;
- adapted procedures independently;
- handled ambiguity with relatively little difficulty.

Average Students

These students generally:

- could apply formulas once demonstrated;
- required substantial practice;
- occasionally confused procedures;
- experienced difficulty in unfamiliar situations.

Lower-Achieving Students

These students frequently:

- relied heavily on memorization;
- struggled with formula selection;
- exhibited weak geometric intuition;
- experienced substantial difficulty with ambiguous-case reasoning.

The effectiveness of any pedagogical sequence must therefore be evaluated across the full range of learners rather than solely among the strongest students.

7.3 Implementation of the Alternative Sequence

7.3.1 Initial Lessons

Under the proposed framework, instruction begins with geometric meaning rather than direct formula application.

Instead of immediately presenting the sine formula, students explore:

- area relationships;
- heights of triangles;
- geometric decomposition;
- trigonometric ratios already learned in junior forms.

This phase serves two purposes.

First, it activates prior knowledge.

Second, it reduces the perception that trigonometric formulas emerge "from nowhere."

Students frequently report that formulas become easier to remember once they understand their origins.

7.3.2 Derivation of the Sine Formula

An important feature of the alternative sequence involves deriving the sine formula through the area formula

$$\text{Area} = \frac{1}{2}ab \sin C$$

rather than presenting it as a standalone theorem.

Classroom observations suggest several advantages.

Students often:

- ask more questions concerning why formulas work;
- recognise relationships between geometry and algebra;
- develop greater confidence during subsequent applications.

Rather than viewing the sine formula as an isolated rule, they perceive it as a direct consequence of geometric reasoning.

Such observations are consistent with constructivist principles discussed previously.

7.3.3 Introduction of the Cosine Formula

The cosine formula was introduced as an extension of the Pythagorean theorem.

For many students, this proved helpful because:

- Pythagoras' theorem was already familiar;
- the derivation appeared logical;
- conceptual continuity was maintained.

Students were more willing to accept the cosine formula when it was framed as a generalization of existing knowledge.

This finding highlights the importance of connecting new concepts with prior learning.

7.4 Student Responses to Different Formulae

7.4.1 Reactions to the Sine Formula

Many students initially regarded the sine formula as easier than the cosine formula.

Several reasons emerged:

- simpler symbolic structure;
- direct substitutions;
- fewer algebraic manipulations;
- quicker calculator execution.

Consequently, students often attempted to use the sine formula whenever possible.

However, this preference occasionally resulted in overreliance.

Some students failed to recognize situations in which the cosine formula offered significant advantages.

This phenomenon suggests that ease of use does not necessarily lead to strategic competence.

7.4.2 Reactions to the Cosine Formula

Students frequently described the cosine formula as:

- longer;



- more complicated;
- harder to remember.

Yet an interesting pattern emerged.

Once students understood the derivation and purpose of the formula, their attitudes became more positive.

Many began to recognize advantages such as:

- exact-value solutions;
- elimination of ambiguity;
- direct determination of unknown sides.

This shift suggests that negative attitudes toward the cosine formula may stem partly from instructional presentation rather than inherent mathematical difficulty.

7.5 Classroom Analysis of the Ambiguous Case

7.5.1 Traditional Difficulties

The ambiguous case remains one of the most challenging topics in school trigonometry.

Traditional instruction often begins by dividing situations into numerous categories.

Students must determine whether:

- no triangle exists;
- one triangle exists;
- two triangles exist.

The original reflection paper observed that average learners frequently found this classification process confusing.

This observation was repeatedly confirmed through classroom interactions.

7.5.2 Delayed Introduction

Within the alternative sequence, the ambiguous case is introduced later.

By this stage students have already learned:

- area formulae;
- sine formula;
- cosine formula;
- geometric interpretation;
- strategic method selection.

Consequently, they possess a richer conceptual framework.

Teachers observed that students appeared more receptive to the idea of multiple solutions than when the topic was introduced immediately after learning the sine formula.

7.5.3 Student Interpretations

Even with delayed instruction, misconceptions remained.

Common observations included:

Misconception 1



The calculator gives the only possible answer.

Misconception 2

Triangles always possess exactly one solution.

Misconception 3

Obtuse angles are automatically incorrect.

Misconception 4

A numerical answer must always correspond to a valid triangle.

These misconceptions suggest that ambiguity is fundamentally challenging and cannot be completely eliminated through sequencing alone.

However, delayed introduction appears to make the topic more manageable.

7.6 Examination Script Analysis

7.6.1 Importance of Error Analysis

One of the most informative sources of evidence comes from student examination scripts.

Errors often reveal:

- misconceptions;
- gaps in understanding;
- procedural weaknesses;
- conceptual misunderstandings.

Rather than simply recording incorrect answers, teachers can use such errors to evaluate instructional effectiveness.

7.6.2 Case Study One: Failure to Recognize the Obtuse Angle

The original paper presents a question involving an isosceles triangle in which students applied the sine formula to determine an angle.

Many students obtained:

55.0243°

and immediately accepted this value.

However, the geometric context required the supplementary angle:

124.976°

The error was not computational.

Students applied the formula correctly.

The difficulty arose because they neglected geometric interpretation.

7.6.3 Educational Implications

This error reveals a fundamental tension in trigonometry instruction.

Students often view problem solving as:

1. choose formula;
2. substitute values;

3. calculate answer.

Missing from this process is a critical step:

1. evaluate geometric reasonableness.

The alternative pedagogical framework attempts to address this issue through continuous emphasis on geometric reasoning.

7.7 Case Study Two: The Lighthouse Problem

7.7.1 Overview of the Task

The second examination example involved bearings, distances, and triangle geometry. Students were asked to determine whether one distance exceeded another.

Many students approached the problem by:

- calculating one distance;
- calculating the second distance;
- comparing answers.

This method is correct but computationally intensive.

7.7.2 An Alternative Student Solution

One student solved the problem differently.

Instead of performing extensive calculations, the student observed:

- one angle measured 85° ;
- another measured 25° ;
- the side opposite the larger angle must be longer.

Therefore:

$$AL > BL$$

and John's claim was correct.

7.7.3 Significance of the Solution

This solution is pedagogically important because it demonstrates:

- conceptual understanding;
- efficiency;
- geometric reasoning;
- strategic competence.

The student recognized that trigonometric formulas were not necessary.

The underlying geometric principle provided a simpler path to the answer.

Such solutions exemplify the type of mathematical thinking encouraged by the alternative framework.

7.8 Perceived Benefits of the Alternative Framework

7.8.1 Improved Conceptual Coherence

One frequently observed benefit was greater conceptual connectedness.

Students increasingly recognized relationships among:

- area formulae;
- sine formulae;
- cosine formulae;
- geometric properties.

This contrasts with the fragmented learning often associated with formula-centred instruction.

7.8.2 Enhanced Geometric Awareness

Students appeared more likely to:

- inspect diagrams carefully;
- estimate reasonable answers;
- identify impossible solutions;
- interpret angles meaningfully.

Although such improvements are difficult to quantify, they were evident during classroom discussions.

7.8.3 Better Method Selection

Students gradually became more comfortable asking:

- Which method is easiest?
- Which method gives exact values?
- Which method avoids ambiguity?

Such questions indicate developing strategic competence.

7.8.4 Reduced Formula Dependence

An interesting outcome was a reduction in excessive formula reliance.

Students became more willing to utilize:

- angle properties;
- triangle properties;
- geometric observations;
- mathematical reasoning.

The development of such flexibility is one of the principal aims of mathematics education.

7.9 Challenges Encountered

7.9.1 Time Constraints

The alternative framework generally requires more instructional time.

Activities involving:

- derivations;
- discussions;
- comparisons;
- reflection;



consume more classroom time than direct procedural instruction.

In examination-oriented environments, this may create practical tensions.

7.9.2 Teacher Preparation

Successful implementation requires teachers to possess strong conceptual knowledge.

Teachers must be comfortable discussing:

- alternative methods;
- multiple solutions;
- geometric reasoning;
- derivations.

Professional development may therefore be necessary.

7.9.3 Student Expectations

Some students initially resist deeper conceptual exploration.

Many have become accustomed to:

- receiving formulas;
- memorizing procedures;
- practising exercises.

When confronted with derivations and discussions, they may perceive instruction as more demanding.

Over time, however, many students begin to appreciate the resulting understanding.

7.10 Preliminary Evaluation of the Alternative Sequence

Based on reflective classroom observations, the alternative sequence appears to provide several educational advantages.

Strengths

- Greater conceptual coherence.
- Stronger geometric reasoning.
- Improved awareness of mathematical connections.
- Enhanced strategic thinking.
- Better appreciation of the roles of both formulas.
- Reduced confusion during early learning.

Remaining Difficulties

- Ambiguity remains challenging.
- Additional instructional time is required.
- Some students continue to favour procedural approaches.
- Formal empirical validation remains necessary.

Thus, the framework should be viewed as promising rather than conclusively proven.

7.11 Summary



This chapter has presented a preliminary classroom assessment of the proposed pedagogical sequence based upon reflective teaching experiences in a Hong Kong secondary school. Drawing upon classroom observations and examination analyses from the original reflection paper, the discussion suggests that the alternative framework may support stronger conceptual understanding, improved geometric reasoning, and increased strategic competence among learners.

Several key findings emerged:

1. Deriving formulas promotes conceptual understanding.
2. Strong geometric foundations improve later problem solving.
3. Delaying the ambiguous case may reduce cognitive burden.
4. Students benefit from comparing alternative methods.
5. Examination errors frequently arise from conceptual rather than computational weaknesses.
6. Strategic reasoning can sometimes outperform formula-based calculations.

While these observations are necessarily preliminary and qualitative, they provide encouraging evidence that alternative instructional sequences deserve further consideration. The next chapter therefore expands the discussion through a detailed analysis of student errors, misconceptions, and mathematical thinking processes, using these insights to illuminate broader issues in trigonometry education.

Chapter 8

Error Analysis, Student Misconceptions, and Mathematical Thinking in Solving Triangles

8.1 Introduction

Understanding student errors is an essential component of effective mathematics teaching. While incorrect answers are often viewed as indicators of failure, contemporary mathematics education research suggests that errors should instead be regarded as valuable windows into students' thinking processes. Errors reveal how learners interpret mathematical concepts, apply procedures, construct mental models, and make decisions during problem solving.

In trigonometry, student errors are particularly informative because successful problem solving requires the integration of several forms of knowledge simultaneously:

- geometric understanding;
- algebraic manipulation;
- trigonometric reasoning;
- procedural fluency;
- calculator competency;
- strategic decision-making.

Consequently, mistakes often arise not from a single misconception but from interactions among multiple conceptual and procedural factors.

The original reflection paper identified several recurring errors observed during classroom teaching and examination marking. These observations provide a useful foundation for examining broader patterns of student thinking in the topic of solving triangles.

This chapter analyses common misconceptions associated with the sine formula and cosine formula, explores the underlying cognitive mechanisms that produce these errors, and considers how the alternative pedagogical framework proposed in Chapter 6 may help address them.

8.2 Why Error Analysis Matters

8.2.1 Errors as Evidence of Thinking

A traditional view of assessment treats errors as deficits.

From this perspective:

- correct answers indicate learning;
- incorrect answers indicate failure.

However, educational research increasingly recognizes that student work often reveals much more than a simple distinction between right and wrong.

For example, two students may produce the same incorrect answer for completely different reasons:

- one may misunderstand the concept;
- one may perform a calculation incorrectly.

Effective teaching therefore requires identification of the underlying source of the error.

8.2.2 The Diagnostic Function of Errors

Errors can help teachers identify:

Conceptual Difficulties

Students may not understand the mathematical relationships involved.

Procedural Difficulties

Students may know what to do but fail to execute procedures correctly.

Strategic Difficulties

Students may choose inappropriate methods.

Metacognitive Difficulties

Students may fail to evaluate the reasonableness of their answers.

This classification is particularly useful when analysing trigonometry learning.

8.3 A Framework for Classifying Trigonometric Errors

Based on classroom observations and existing literature, errors in solving triangles may be classified into five broad categories:

Category 1

Geometric Interpretation Errors

Category 2

Formula Selection Errors

Category 3

Ambiguity Errors

Category 4

Calculator and Computational Errors

Category 5

Reasoning and Strategy Errors

These categories provide a framework for analysing student thinking.

8.4 Geometric Interpretation Errors**8.4.1 Weak Diagram Reading Skills**

One of the most common sources of difficulty involves interpreting geometric diagrams.

Many students approach trigonometry questions as algebraic exercises rather than geometric situations.

Typical behaviours include:

- ignoring relationships shown in diagrams;
- overlooking angle properties;
- neglecting symmetry;
- failing to identify special triangles.

Students often focus immediately on formula selection without first understanding the geometry.

8.4.2 Loss of Geometric Meaning

This problem becomes particularly evident when students view trigonometric formulas purely as computational tools.

For instance, after obtaining an angle value from a calculator, students frequently accept the result automatically without asking:

- Does this angle fit the diagram?
- Should the angle be acute or obtuse?
- Is the triangle reasonable?

Such behaviours suggest that geometric interpretation has become disconnected from calculation.

8.4.3 The Examination Example

The original reflection paper documented a question involving an isosceles triangle in which students had to determine the size of an angle using the sine formula. Many students correctly obtained:

55.0243°

but failed to recognize that the required angle should be:

124.976°

because the geometry of the figure required an obtuse angle.

This error is particularly informative.

Students successfully performed the trigonometric calculation.

The failure occurred during interpretation.

In other words, the difficulty was not procedural but conceptual.

8.4.4 Educational Implications

This type of error suggests that classroom instruction should emphasize:

- diagram analysis;
- geometric estimation;
- visual reasoning;
- interpretation of solutions.

Students must learn that solving a trigonometric equation represents only part of the problem-solving process.

8.5 Formula Selection Errors

8.5.1 Confusion Between Sine and Cosine Formulae

Many students struggle to determine which formula should be applied.

Typical classroom questions include:

- Should I use the sine formula?
- Should I use the cosine formula?
- Can either formula be used?

Such uncertainty reflects incomplete understanding of the structure of triangle-solving problems.

8.5.2 Surface-Level Recognition

Novice learners often select formulas based on superficial characteristics.

For example:

- seeing two sides and one angle;
- noticing a familiar diagram shape;
- recalling a recently taught example.

These approaches may work for routine exercises but frequently fail when problems differ slightly from textbook examples.

8.5.3 Absence of Strategic Understanding

Expert problem solvers typically consider:

- available information;
- desired outcomes;
- computational efficiency;
- possibility of exact solutions;
- potential ambiguities.

Students, however, often focus exclusively on obtaining an answer.

Consequently, formula selection becomes mechanical rather than strategic.

8.5.4 Pedagogical Implications

The findings support an important principle of the alternative pedagogical framework:

Students should compare methods explicitly rather than learning formulas in isolation.

Such comparisons promote strategic competence and deeper understanding.

8.6 Ambiguity Errors and the S.S.A. Case

8.6.1 The Most Persistent Trigonometric Misconception

Among all topics in solving triangles, the ambiguous case appears to generate the greatest number of conceptual errors.

The difficulty arises because students frequently assume that every mathematical question has exactly one answer.

However, S.S.A. situations may produce:

- no (real) solution;
- one solution;
- two solutions.

For many learners, this is a fundamentally unfamiliar situation.

8.6.2 Misunderstanding the Inverse Sine Function

Students often interpret the calculator operation

$\sin^{-1}(x)$

as producing the unique angle associated with the sine value.

For example,

$$\sin^{-1}(0.75) = 48.59037789^\circ$$

may be viewed as the only possible answer.

Students therefore overlook the supplementary angle

$$131.4096221^\circ$$

which possesses the same sine value.

This misconception was explicitly identified in the original reflection paper.

8.6.3 Overgeneralization of Calculator Outputs

Calculators provide a principal value.

Students frequently assume:

calculator answer = mathematical answer

This assumption is understandable but problematic.

While calculators generate numerical outputs correctly, interpreting those outputs remains the responsibility of the learner.

8.6.4 Failure to Consider Multiple Geometric Configurations

Another difficulty involves visualization.

Students may struggle to imagine that two different triangles can satisfy the same conditions.

As a result, the existence of multiple solutions appears counterintuitive.

The challenge is therefore not merely computational but spatial and conceptual.

8.6.5 Educational Responses

Effective instructional strategies may include:

- dynamic geometry software;
- physical models;
- multiple diagram representations;
- delayed introduction of ambiguity;
- explicit discussion of inverse functions.

These approaches may strengthen understanding of why ambiguity occurs.

8.7 Computational and Calculator Errors

8.7.1 Mechanical Mistakes

Some errors are purely computational.

Examples include:

- incorrect substitutions;
- sign errors;
- arithmetic mistakes;
- rounding errors.



Although important, such errors are generally easier to diagnose and address.

8.7.2 Calculator Mode Errors

Students frequently forget to check whether calculators are operating in:

- degree mode;
- radian mode.

This error can produce dramatically incorrect answers.

Because students often trust calculator outputs, these mistakes may remain undetected.

8.7.3 Blind Trust in Technology

A more significant issue involves overdependence on technology.

Some students:

- rarely estimate answers;
- fail to verify results;
- accept outputs uncritically.

This behaviour weakens mathematical judgment.

Technology should support thinking rather than replace it.

8.8 Strategy Errors and Non-Efficient Solutions

8.8.1 When Correct Solutions Reflect Weak Understanding

Not all errors result in incorrect answers.

Sometimes students obtain correct answers through unnecessarily complicated methods.

Such solutions may reveal weaknesses in strategic thinking.

8.8.2 The Lighthouse Problem

A particularly insightful example appears in the original reflection paper.

Students were asked to determine whether one distance exceeded another.

Many students calculated both distances using the sine formula before making a comparison.

However, one student recognized that:

- the larger angle is opposite the longer side.

Therefore:

$$AL > BL$$

without extensive calculation.

8.8.3 What This Reveals About Mathematical Expertise

The solution demonstrates characteristics commonly associated with mathematical expertise:

Efficiency

Avoiding unnecessary calculations.

Conceptual Understanding

Applying fundamental geometric principles.

Flexibility

Recognizing alternative solution paths.

Strategic Competence

Selecting an effective approach.

These qualities are central objectives of mathematics education.

8.9 Student Thinking Through the Lens of APOS Theory

The error patterns discussed above may be interpreted through APOS Theory.

8.9.1 Action-Level Thinking

Students at the Action level:

- substitute values mechanically;
- follow learned procedures;
- rely heavily on memorization.

Many ambiguity-related errors occur at this stage.

8.9.2 Process-Level Thinking

Students begin recognizing why procedures work.

They can:

- interpret diagrams;
- explain calculations;
- anticipate outcomes.

Errors become less frequent.

8.9.3 Object-Level Thinking

Students treat formulas as coherent mathematical objects.

They compare:

- methods;
- advantages;
- limitations.

Strategic reasoning becomes evident.

8.9.4 Schema-Level Thinking

Students develop integrated understanding.

Connections emerge among:

- geometry;
- algebra;
- trigonometry;
- mathematical reasoning.

The strongest student solutions often display Schema-level understanding.

8.10 Student Thinking Through the Lens of Cognitive Load Theory

8.10.1 High Cognitive Demands of Triangle Solving

When solving triangles, students must coordinate many elements simultaneously:

- formulas;
- correspondences;
- diagrams;
- calculations;
- interpretations.

Working memory may therefore become overloaded.

8.10.2 Ambiguity as a Source of Cognitive Burden

The ambiguous case requires students to consider multiple possibilities simultaneously.

This places considerable demands on working memory.

Consequently, students may simplify the task by accepting the first answer generated by a calculator.

While understandable, this shortcut often produces errors.

8.10.3 Reducing Cognitive Overload

The alternative pedagogical framework attempts to reduce cognitive burden by:

- delaying ambiguity;
- emphasizing geometric meaning;
- strengthening conceptual links;
- promoting strategic reasoning.

Such measures may free cognitive resources for genuine understanding.

8.11 Implications for Classroom Teaching

Analysis of student errors suggests several instructional priorities.

Priority 1: Strengthen Geometric Reasoning

Students should learn to interpret diagrams before applying formulas.

Priority 2: Emphasize Meaning Rather Than Memorization

Formula derivations and conceptual connections should be highlighted.

Priority 3: Encourage Multiple Solution Methods

Students should compare efficiency and appropriateness.

Priority 4: Develop Metacognitive Habits

Learners should routinely ask:

- Does my answer make sense?
- Is there another solution?
- Can I verify this result?

Priority 5: Use Errors as Learning Opportunities

Incorrect solutions should be analysed and discussed openly.

Error analysis can become a powerful tool for conceptual development.

8.12 Toward a Model of Mathematical Sense-Making

The cumulative evidence presented throughout this chapter suggests that successful trigonometry learning requires more than procedural competence.

Students must learn to integrate:

- geometric sense;
- procedural fluency;
- conceptual understanding;
- strategic competence;
- reflective judgment.

These components can be viewed collectively as mathematical sense-making.

The ultimate goal is not merely to solve triangles correctly but to understand why solutions are correct and when alternative approaches may be preferable.

8.13 Summary

This chapter examined student errors, misconceptions, and mathematical thinking associated with solving triangles using the sine and cosine formulae. Drawing upon classroom experiences reported in the original reflection paper, the analysis demonstrated that many student difficulties arise from conceptual and strategic weaknesses rather than computational deficiencies.

Several key findings emerged:

1. Geometric interpretation errors are common.
2. Formula selection often reflects procedural rather than conceptual thinking.
3. The ambiguous case remains a major source of misconceptions.
4. Calculator outputs are frequently accepted without critical evaluation.
5. Efficient solutions often arise from geometric reasoning rather than extensive computation.
6. Student errors reveal valuable information about cognitive development.
7. Mathematical sense-making should be regarded as a central goal of trigonometry education.

The alternative pedagogical framework proposed in this manuscript seeks to address these issues through stronger conceptual connections, enhanced geometric reasoning, strategic method comparison, and reduced cognitive overload. The next chapter examines the broader implications of these findings for curriculum development, textbook design, teacher education, assessment practices, and mathematics education policy in Hong Kong and beyond.

Chapter 9

Implications for Curriculum Development, Textbook Design, Teacher Education, and Assessment

9.1 Introduction

The preceding chapters have examined the historical development of trigonometry, reviewed relevant literature, established a theoretical framework, critiqued the conventional pedagogical sequence, proposed an alternative instructional model, and analyzed classroom observations and student thinking. Collectively, the findings suggest that the order in which trigonometric formulae are introduced may significantly influence students' conceptual understanding, procedural fluency, strategic competence, and confidence.

The purpose of this chapter is to explore the broader educational implications of these findings. Although the present study is conceptual rather than empirical, the arguments developed throughout the manuscript raise important questions concerning curriculum design, textbook development, classroom instruction, teacher professional development, and assessment practices.

In many educational systems, discussions about mathematics education focus primarily on curriculum content. Considerable attention is devoted to determining what students should learn, while relatively less consideration is given to the sequencing through which learning occurs. The present study argues that sequence itself should be viewed as an important curriculum variable deserving deliberate attention.

This chapter therefore considers how the alternative pedagogical framework proposed in Chapter 6 may inform future developments within secondary mathematics education, particularly in the context of Hong Kong.

9.2 Implications for Curriculum Development

9.2.1 Curriculum as a Learning Journey

Curriculum documents often describe mathematical knowledge as a collection of learning outcomes.

Students are expected to:

- learn concepts;
- master procedures;
- solve problems;
- demonstrate understanding.

However, these outcomes do not specify the pathways through which understanding develops.

From the perspective of the present study, curriculum should be viewed not merely as a list of topics but as a carefully organized learning journey.

The sequence through which ideas are encountered may significantly affect:

- cognitive accessibility;
- conceptual coherence;
- long-term retention;
- transfer of learning.

Consequently, curriculum developers should consider not only what is taught but also when and how concepts are introduced.

9.2.2 Rethinking the Position of the Ambiguous Case



One of the most significant implications of this study concerns the placement of the S.S.A. ambiguous case.

In many curricula, the ambiguous case appears immediately after the introduction of the sine formula. This arrangement reflects mathematical classification rather than learning progression. As argued in the original reflection paper, the resulting complexity can overwhelm many learners and generate unnecessary confusion.

Curriculum developers may therefore wish to consider alternative arrangements.

Possible approaches include:

Option A

Retain the traditional sequence.

Option B

Introduce the ambiguous case after students have learned both the sine and cosine formulae.

Option C

Reserve the ambiguous case for enrichment and extension activities.

The alternative framework proposed in this manuscript aligns most closely with Option B.

9.2.3 Emphasizing Connections Across Topics

A recurring theme throughout this study has been the importance of mathematical connectedness.

Students often encounter:

- area formulae;
- sine formulae;
- cosine formulae;
- congruence conditions;
- triangle properties

as separate topics.

Yet these ideas are fundamentally interconnected.

Future curriculum frameworks should place greater emphasis on helping students recognize these relationships.

Such integration may improve:

- conceptual understanding;
- method selection;
- transferability of knowledge.

9.2.4 Balancing Breadth and Depth

Modern curricula frequently face the challenge of overcrowding.

Pressure exists to include:

- more content;
- more applications;
- more technology;
- more assessment objectives.

As a result, instructional time may be fragmented.



The findings of this study suggest that deeper conceptual development may sometimes be more valuable than broader coverage.

Students who understand fewer ideas deeply may ultimately achieve more than students who memorize numerous procedures superficially.

9.3 Implications for Textbook Design

9.3.1 The Influence of Textbooks

In many classrooms, textbooks effectively define the curriculum that students experience.

Teachers frequently determine:

- lesson structure;
- examples;
- exercises;
- assessments

based largely upon textbook organization.

Consequently, textbook sequencing can have significant educational consequences.

9.3.2 Beyond Formula-Centred Organization

A common feature of trigonometry textbooks is organization around formulas.

Typical headings include:

- The Sine Formula
- Applications of the Sine Formula
- The Cosine Formula
- Applications of the Cosine Formula

While logical, such structures may inadvertently reinforce fragmentation.

Students may perceive each formula as an isolated chapter rather than part of a broader mathematical system.

Future textbook designs could instead emphasize conceptual relationships.

For example:

Unit 1

Understanding Triangle Relationships

Unit 2

Area and Trigonometric Connections

Unit 3

Methods for Solving Triangles

Unit 4

Comparing Alternative Strategies

Such structures may encourage a more connected view of mathematics.

9.3.3 Increased Use of Derivations

Many textbooks present formulas as completed products.



Students are expected to memorize and apply them.

However, the original reflection paper demonstrated how the sine formula can be derived naturally from area relationships.

Textbooks may therefore benefit from:

- greater emphasis on derivation;
- visual explanations;
- conceptual development;
- mathematical reasoning.

Such approaches help students understand why formulas work rather than merely how to use them.

9.3.4 Comparative Solution Approaches

Textbooks often present one preferred solution method.

Alternative methods may receive little attention.

The present study suggests that students benefit from comparing:

- sine-rule solutions;
- cosine-rule solutions;
- geometric arguments;
- exact-value methods;
- calculator-based methods.

Systematic comparison helps learners develop strategic competence.

Future textbooks could therefore include dedicated sections entitled:

Comparing Methods

or

Which Approach Is Most Efficient?

Such features would encourage reflective mathematical thinking.

9.4 Implications for Classroom Teaching

9.4.1 Teaching for Understanding

Perhaps the most important implication for teachers concerns the need to prioritize understanding over memorization.

Students frequently ask:

Which formula should I use?

Teachers may wish to encourage a different question:

Why does this formula work?

The shift from procedural instruction to conceptual exploration can promote more durable learning.

9.4.2 Encouraging Mathematical Communication

Mathematics classrooms often reward answers rather than explanations.

Yet the development of understanding frequently depends upon discussion.

Teachers can encourage students to explain:

- why a method works;
- why a solution is reasonable;
- why one strategy may be preferable.

Such conversations help make thinking visible.

9.4.3 Promoting Multiple Methods

One of the most powerful findings emerging from the original reflection paper was the example of a student who solved a comparison problem using the relationship between angle size and side length rather than performing extensive calculations.

This example highlights the importance of encouraging multiple representations and multiple solution strategies.

Teachers can foster such thinking by regularly asking:

- Is there another method?
- Is there a shorter method?
- Is there a more elegant method?

These questions help students move beyond procedural routine.

9.4.4 Using Errors Productively

Traditional classrooms often focus on avoiding mistakes.

However, Chapter 8 demonstrated that student errors can serve as valuable learning opportunities.

Teachers may wish to incorporate:

- error analysis tasks;
- discussion of misconceptions;
- comparison of incorrect and correct solutions.

Such activities can deepen understanding and strengthen metacognitive awareness.

9.5 Implications for Teacher Education

9.5.1 The Need for Strong Conceptual Knowledge

Implementation of the proposed pedagogical framework requires teachers to possess more than procedural competence.

Teachers must understand:

- derivations;
- mathematical connections;
- alternative solution methods;
- common misconceptions;
- conceptual foundations.

Without such knowledge, it is difficult to facilitate meaningful classroom discussions.

9.5.2 Pedagogical Content Knowledge

Shulman's concept of pedagogical content knowledge is particularly relevant.

Effective trigonometry teachers require understanding of:

- what students find difficult;
- why misconceptions arise;
- how concepts can be represented effectively;
- which examples reveal important mathematical ideas.

Teacher education programs should therefore devote greater attention to the teaching of trigonometry itself rather than focusing solely on mathematical content.

9.5.3 Reflective Practice

The original paper itself emerged from reflective teaching practice.

Reflection enables teachers to examine:

- student responses;
- instructional effectiveness;
- assessment outcomes;
- pedagogical assumptions.

Professional development programmes should encourage systematic reflection as a means of improving mathematics instruction.

9.6 Implications for Assessment

9.6.1 Assessing More Than Calculation

Many trigonometry assessments emphasize computational accuracy.

While procedural fluency remains important, such assessments may fail to capture:

- conceptual understanding;
- strategic competence;
- mathematical reasoning.

Assessment practices should therefore be broadened.

9.6.2 Assessing Mathematical Reasoning

Tasks may be designed to require students to:

- justify solutions;
- compare methods;
- identify misconceptions;
- explain reasoning.

Such questions can provide deeper insight into student understanding.

9.6.3 Incorporating Multiple Solution Pathways

Assessment tasks frequently imply a single preferred solution.

However, stronger students often discover alternative approaches.

Examiners and teachers should recognize the educational value of:

- geometric arguments;
- logical deductions;
- conceptual shortcuts.

The lighthouse problem discussed earlier provides an excellent example. A purely geometric argument may demonstrate understanding just as effectively as lengthy trigonometric calculations.

9.6.4 Error-Based Assessment

Another promising approach involves asking students to analyse incorrect solutions.

For example, students might examine a worked solution containing a common ambiguity error and explain:

- what went wrong;
- why the error occurred;
- how it should be corrected.

Such tasks support metacognitive development.

9.7 Implications for Educational Policy

9.7.1 Supporting Flexible Curriculum Interpretation

Educational systems often provide curriculum guides that outline required content.

Greater flexibility may be beneficial regarding instructional sequencing.

Allowing teachers some autonomy to adapt sequences according to learner needs may improve educational outcomes.

9.7.2 Balancing Examination Pressures and Understanding

Hong Kong's educational system places considerable emphasis on public examinations.

While examinations play an important role, excessive focus on procedural performance may inadvertently discourage conceptual teaching.

Policymakers should strive to maintain a balance between:

- procedural fluency;
- conceptual understanding;
- problem-solving ability;
- mathematical reasoning.

9.7.3 Supporting Innovation

Innovative pedagogical approaches require institutional support.

Schools should encourage:

- action research;
- lesson study;
- collaborative planning;
- professional dialogue.

Such activities allow teachers to evaluate alternative approaches systematically.

9.8 Implications Beyond Hong Kong

Although this study is situated within the Hong Kong educational context, many of its conclusions possess broader relevance.

Across different countries, mathematics teachers commonly report similar difficulties involving:

- trigonometric ambiguity;
- formula memorization;
- method selection;
- conceptual understanding.

Therefore, the fundamental questions addressed by this manuscript are not uniquely local.

The broader educational principle may be stated as follows:

Mathematical sequencing should be informed not only by mathematical logic but also by how students learn.

This principle has potential applications beyond trigonometry and beyond Hong Kong.

9.9 A Re-conceptualization of Trigonometry Teaching

The cumulative argument of this manuscript suggests a shift in perspective.

Traditional teaching often views trigonometry as:

- a collection of formulas;
- a collection of procedures;
- a collection of examination techniques.

The alternative framework instead views trigonometry as:

- a network of relationships;
- a system of connected ideas;
- a context for mathematical reasoning.

This re-conceptualization changes the role of both teacher and student.

Teacher

From instructor of procedures to facilitator of mathematical understanding.

Student

From performer of calculations to constructor of meaning.

Such a shift aligns with broader trends in contemporary mathematics education.

9.10 Summary

This chapter has explored the implications of the present study for curriculum development, textbook design, classroom teaching, teacher education, assessment, and educational policy.

Several key recommendations emerged:

1. Curriculum developers should reconsider the sequencing of trigonometric concepts.
2. Textbooks should emphasize conceptual connections alongside procedural techniques.
3. Teachers should promote reasoning, discussion, and strategic thinking.



4. Teacher education should strengthen pedagogical content knowledge in trigonometry.
5. Assessments should evaluate understanding in addition to computational proficiency.
6. Educational policy should support flexibility, innovation, and reflective practice.

Taken together, these implications suggest that improving trigonometry education requires more than introducing new formulas or technologies. Rather, it requires careful attention to how mathematical ideas are organized, connected, and experienced by learners.

The final chapter synthesizes the major findings of the study, revisits the central argument concerning pedagogical sequencing, discusses limitations, and proposes directions for future research regarding the teaching and learning of trigonometric formulae in secondary mathematics education.

Chapter 10

Conclusion and Future Directions

10.1 Introduction

This manuscript began with a seemingly straightforward question arising from classroom practice:

Is the conventional sequence used for teaching the sine formula and cosine formula necessarily the best sequence for student learning?

Although mathematics textbooks in Hong Kong and many other educational systems commonly introduce the sine formula through A.S.A. and A.A.S. cases, followed by the ambiguous S.S.A. case and subsequently the cosine formula, the original reflection paper argued that such a sequence may not be pedagogically optimal for all learners.

The purpose of this study has therefore been to critically examine the pedagogical sequencing of trigonometric formulae, drawing upon classroom reflections, educational theories, historical developments, and contemporary research on mathematics learning. Rather than proposing new mathematical content, the study has focused on a different but equally important question: how existing content might be organized to better support student understanding.

Throughout the manuscript, the sine formula and cosine formula have been treated not merely as computational tools but as interconnected mathematical ideas embedded within a broader conceptual structure. The analysis has suggested that instructional sequence influences not only what students learn, but also how they understand, retain, and apply that knowledge.

This final chapter synthesizes the major findings of the study, revisits the central arguments, identifies limitations, and proposes directions for future research and practice.

10.2 Revisiting the Purpose of the Study

The study was guided by five overarching questions:

1. What assumptions underlie the conventional sequence for teaching trigonometric formulae?
2. What learning difficulties emerge under the traditional approach?
3. How can educational theories inform the design of alternative instructional pathways?
4. What advantages may arise from an alternative pedagogical sequence?
5. What implications follow for curriculum development and mathematics education?

To address these questions, the manuscript examined trigonometry from multiple perspectives:

- historical;
- theoretical;
- pedagogical;
- curricular;
- classroom-based.

The resulting analysis indicates that the sequencing of mathematical ideas deserves much greater attention than it currently receives within curriculum discussions.

10.3 Summary of Major Findings

10.3.1 Finding 1: Historical Development Does Not Necessarily Support Current Textbook Sequences

Chapter 2 demonstrated that trigonometry evolved through a long historical process involving contributions from ancient Egyptian, Babylonian, Greek, Indian, Islamic, and European mathematical traditions.

Importantly, the historical development of trigonometry did not follow the sequence commonly adopted in modern textbooks.

Concepts emerged gradually through:

- practical measurement problems;
- astronomical investigations;
- geometric reasoning;
- algebraic generalization.

This observation suggests that current textbook structures are not inevitable consequences of mathematics itself. Rather, they represent historical and curricular choices that may legitimately be reconsidered.

10.3.2 Finding 2: Solving Triangles Is Conceptually Complex

The literature reviewed in Chapter 3 demonstrated that trigonometry presents substantial challenges for learners.

Students must coordinate:

- geometric representations;
- algebraic representations;
- trigonometric functions;
- calculator technology;
- mathematical reasoning.

The difficulties documented in previous research closely mirror those observed in the original reflection paper.

Common challenges include:

- ambiguity;
- formula selection;
- diagram interpretation;
- inverse trigonometric functions;
- strategic decision-making.

These difficulties suggest that careful instructional design is essential.

10.3.3 Finding 3: Educational Theories Support Greater Emphasis on Conceptual Connections

The theoretical framework developed in Chapter 4 highlighted several important principles.

Constructivism

Students learn more effectively when new ideas emerge from existing knowledge structures.

Cognitive Load Theory

Instructional sequences should minimize unnecessary cognitive burden.

APOS Theory

Learning progresses from actions to integrated conceptual schemas.

Variation Theory

Comparisons and contrasts help reveal important mathematical features.

Relational Understanding

Deep learning depends upon understanding relationships among concepts.

Taken collectively, these theories strongly support an instructional approach emphasizing conceptual connections rather than isolated procedures.

10.3.4 Finding 4: The Conventional Sequence Possesses Both Strengths and Weaknesses

Chapter 5 acknowledged several advantages of the traditional approach:

- familiarity;
- mathematical coherence;
- examination alignment;
- simplicity of organization.

However, significant limitations were also identified.

Most notably:

- the ambiguous case is introduced relatively early;
- formulas are often treated independently;
- strategic competence receives limited attention;
- geometric reasoning may be overshadowed by procedural computation.

The analysis suggests that many commonly observed student errors may be linked to these instructional characteristics.

10.3.5 Finding 5: An Alternative Sequence Offers Pedagogical Promise

Chapter 6 proposed an alternative framework in which:

1. geometric foundations are revisited;
2. area formulas are introduced first;
3. the sine formula is derived rather than presented;
4. the cosine formula is introduced as a generalization of Pythagoras' theorem;
5. strategic comparison of methods is encouraged;
6. the ambiguous case is delayed until later stages.

The proposed sequence aims to strengthen:

- conceptual understanding;
- mathematical connectedness;
- strategic reasoning;
- learner confidence.

Rather than replacing existing content, the framework reorganizes content to support learning more effectively.

10.3.6 Finding 6: Classroom Reflections Suggest Positive Outcomes

Chapter 7 examined classroom experiences associated with the alternative approach.

Several encouraging observations emerged.

Students appeared more likely to:

- recognize conceptual connections;
- discuss mathematical reasoning;
- compare methods strategically;
- interpret diagrams carefully.

Although these observations remain preliminary and qualitative, they suggest that alternative sequencing may be worthy of further investigation.

10.3.7 Finding 7: Student Errors Reveal Conceptual Rather Than Computational Difficulties

The error analyses conducted in Chapter 8 indicated that many student difficulties arise not from calculation errors but from deeper misconceptions.

Common problems included:

- failure to interpret diagrams;
- overreliance on calculator outputs;
- misunderstanding of ambiguity;
- mechanical formula application.

These findings reinforce the importance of emphasizing understanding rather than merely procedural execution.

10.3.8 Finding 8: Implications Extend Beyond Trigonometry

The discussion in Chapter 9 demonstrated that the issues explored in this study have broader relevance.

Questions concerning:

- sequencing;
- conceptual development;
- cognitive load;
- strategic competence;

are not unique to trigonometry.

They apply across many areas of mathematics education.

Consequently, the present study contributes to wider discussions concerning curriculum design and pedagogical decision-making.

10.4 The Central Argument of the Study

A central claim has emerged repeatedly throughout this manuscript:

The pedagogical sequence of mathematical ideas should be determined not only by mathematical logic but also by how students learn.

Traditional textbook sequences are often designed from the perspective of mathematics.

The alternative framework proposed here begins instead from the perspective of learners.

This distinction is important.

What appears simple to mathematicians may not appear simple to students.

For example:

- introducing the sine rule before the cosine rule may be mathematically logical;
- introducing the ambiguous case immediately afterwards may be mathematically convenient;

but neither decision is automatically pedagogically optimal.

The present study therefore advocates a learner-centred view of curriculum organization.

10.5 Contributions of the Study

The study makes several contributions to mathematics education.

10.5.1 A Conceptual Reinterpretation of Solving Triangles

Rather than viewing solving triangles as a collection of formulas, the study presents it as a network of interconnected ideas.

Relationships among:

- area formulas;
- sine formulas;
- cosine formulas;
- geometric principles;

are emphasized throughout.

10.5.2 A Critique of Traditional Sequencing

The manuscript provides a systematic examination of assumptions underlying conventional textbook structures.

Such discussions are relatively uncommon in trigonometry education literature.

10.5.3 An Alternative Pedagogical Framework

A coherent alternative sequencing model has been developed and justified through theory and classroom reflection.

10.5.4 Integration of Theory and Practice

The study illustrates how educational theories can inform practical instructional decisions.

This integration helps bridge the often-observed gap between educational research and classroom teaching.

10.6 Limitations of the Study

Despite its contributions, several limitations should be acknowledged.

10.6.1 Conceptual Rather Than Empirical Nature

The study is primarily conceptual.

Although classroom observations provide valuable insights, the manuscript does not include:

- large-scale quantitative data;
- experimental comparisons;
- controlled intervention studies.

Consequently, conclusions should be interpreted cautiously.



10.6.2 Single Context

The reflections originated within a Hong Kong secondary school context.

Different educational systems may possess:

- different curricula;
- different examinations;
- different teaching traditions.

Further studies are needed to explore broader applicability.

10.6.3 Teacher Perspective

The analysis is grounded largely in teacher observations.

Future research should also examine:

- student perspectives;
- classroom discourse;
- learning trajectories;
- long-term outcomes.

10.6.4 Scope of Content

The study focuses specifically on the topic of solving triangles using the sine and cosine formulae.

Other aspects of trigonometry, such as:

- trigonometric identities;
- graphs of trigonometric functions;
- advanced applications;

remain beyond the scope of the current investigation.

10.7 Directions for Future Research

Several promising directions emerge from this study.

10.7.1 Comparative Classroom Studies

Future researchers may compare:

Traditional Sequence

versus

Alternative Sequence

through quasi-experimental designs.

Possible outcome measures include:

- conceptual understanding;
- procedural fluency;
- retention;
- examination performance.

10.7.2 Student Interviews



Interviews could provide richer insight into:

- students' conceptual development;
- perceptions of ambiguity;
- strategy selection processes;
- mathematical reasoning.

10.7.3 Lesson Study Projects

Collaborative lesson study initiatives may explore how different sequencing approaches function in practice.

Such work could generate valuable evidence for teachers and curriculum developers.

10.7.4 Dynamic Technology Integration

Future studies may investigate how:

- dynamic geometry software;
- interactive simulations;
- digital visualization tools;

support understanding of solving triangles and the ambiguous case.

10.7.5 Cross-Cultural Comparisons

Different educational systems organize trigonometry differently.

Comparative research may identify instructional approaches that support learning across diverse contexts.

10.7.6 Extension to Other Mathematical Topics

The broader principles developed here may be applied to other areas of mathematics where sequencing is important.

Examples include:

- algebraic manipulation;
- calculus;
- probability;
- coordinate geometry.

10.8 Final Reflections on Mathematics Teaching

The origins of this study lie not in large-scale research projects but in ordinary classroom experiences.

Teachers continually observe:

- questions students ask;
- mistakes students make;
- strategies students use;
- moments of understanding.

The original reflection paper emerged from precisely such experiences.



One of the most important lessons arising from this study is that effective mathematics teaching is not simply about transmitting methods.

Students may learn to apply formulas correctly while still misunderstanding the mathematics involved.

Conversely, genuine understanding often becomes visible when students:

- reason geometrically;
- compare alternative approaches;
- question assumptions;
- explain solutions.

It is these habits of mind that mathematics education should ultimately seek to cultivate.

10.9 Concluding Remarks

This manuscript has argued that the teaching of trigonometric formulae deserves reconsideration, particularly with respect to the sequence in which ideas are introduced. Drawing upon reflective practice, educational theory, and research literature, a case has been made for a more connected, conceptually oriented, and learner-centred approach to solving triangles.

The proposed alternative framework does not claim to offer a definitive solution. Rather, it serves as an invitation to teachers, curriculum developers, textbook authors, and researchers to re-examine assumptions that have long been taken for granted.

Ultimately, the question is not whether students can learn the sine formula and cosine formula through traditional methods. Clearly, many do.

The more important question is whether they can learn these ideas more meaningfully, more confidently, and with deeper understanding.

If mathematics education seeks not merely successful examination performance but genuine mathematical thinking, then the sequencing of learning experiences deserves careful and continuing attention.

The present study is offered as a contribution to that ongoing conversation.

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**Appendices****Appendix A****Conventional versus Alternative Pedagogical Sequence****Conventional Sequence**

Stage	Content
1	Sine Formula (A.S.A./A.A.S.)
2	Sine Formula (S.S.A Ambiguous Case)
3	Cosine Formula (S.A.S.)
4	Cosine Formula (S.S.S.)
5	Mixed Applications

Proposed Alternative Sequence

Stage	Content
1	Review of Triangle Geometry
2	Area Formula $\text{Area} = \frac{1}{2}ab \sin C$
3	Derivation of Sine Formula
4	Applications of Sine Formula (A.S.A./A.A.S.)
5	Derivation of Cosine Formula
6	Applications of Cosine Formula (S.A.S./S.S.S.)
7	Comparison of Solution Strategies
8	S.S.A Ambiguous Case
9	Advanced Problem Solving

**Appendix B****Summary of Triangle-Solving Methods**

Given Information	Preferred Method
A.S.A.	Sine Formula
A.A.S.	Sine Formula
S.A.S.	Cosine Formula First
S.S.S.	Cosine Formula First
S.S.A	Sine Formula with Ambiguity Check
Exact Value Required	Cosine Formula Preferred
Method Comparison	Sine and Cosine Formulae



Appendix C

Classroom Discussion Questions

Conceptual Understanding

1. Why does the sine formula work?
2. How is the sine formula related to the area formula?
3. Why is the cosine formula considered a generalization of Pythagoras' Theorem?
4. Why can the sine formula sometimes produce two answers?

Strategic Thinking

1. Which formula would you use first? Why?
2. Is there a shorter method?
3. Can the problem be solved without trigonometry?
4. Is an exact answer possible?

Reflection Questions

1. Does your answer make geometric sense?
2. How can you verify your solution?

**Appendix D****Common Student Misconceptions**

Misconception	Possible Cause
Calculator answer is always correct	Overreliance on technology
Every triangle has exactly one solution	Previous mathematical experience
Sine formula and cosine formula are unrelated	Fragmented instruction
Obtuse-angle solutions are impossible	Weak geometric reasoning
Formula selection depends only on memorized rules	Procedural learning

**Appendix E****Examination Error Analysis Template****Problem**

Student Solution

Error Classification

- ☐ Conceptual Error
- ☐ Procedural Error
- ☐ Formula Selection Error
- ☐ Ambiguity Error
- ☐ Computational Error
- ☐ Strategic Error

Explanation

Correct Reasoning

Suggested Teaching Response

Appendix F**Sample Lesson Sequence (Five Lessons)****Lesson 1**

Review of triangle geometry and area concepts

Lesson 2

Derivation and application of

$$\text{Area} = \frac{1}{2} ab \sin C$$

Lesson 3

Derivation and application of the sine formula

Lesson 4

Introduction and application of the cosine formula

Lesson 5

Comparison of methods and strategic problem solving



Appendix G

Suggested Future Research Questions

1. Does delaying the ambiguous case improve conceptual understanding?
2. How does sequencing influence cognitive load?
3. Which solution method do students prefer after explicit comparison?
4. How does procedural fluency differ under alternative instructional sequences?
5. What role does geometric reasoning play in reducing formula misuse?
6. Does the proposed framework improve long-term retention?